Time Deep Gradient Flow Method for Option Pricing in rough Diffusion Models

SIAM Conference on Financial Mathematics and Engineering

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joint work with Antonis Papapantoleon

July 17, 2025

Pricing

Price of a derivative with pay-off $\Phi(S_T)$

$$u(t, \mathbf{x}) = \mathbb{E}\left[e^{-r(T-t)}\Phi(S_T)|S_t\right]$$

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Feynman-Kac formula:

$$\frac{\partial u}{\partial t} - \sum_{i,j=0}^{n} a^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{i=0}^{n} b^i \frac{\partial u}{\partial x_i} + ru = 0$$

$$u(0, \mathbf{x}) = \Phi(\mathbf{x})$$

Rough Heston

- Not Markovian $\implies$ no PDE

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- Lifted Heston: Markovian, but multiple dimensions

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sqrt{V_t^n}S_t\mathrm{d}W_t \qquad S_0 > 0$$

$$V_t^n = g^n(t) + \sum_{i=1}^{n} c_i^n V_t^{n,i}$$

$$\mathrm{d}V_t^{n,i} = -\left(\gamma_i^n V_t^{n,i} + \lambda V_t^n\right)\mathrm{d}t + \eta\sqrt{V_t^n}\mathrm{d}B_t \quad V_0^{n,i} = 0$$

$$g^n(t) = V_0 + \lambda\theta\sum_{i=1}^{n} c_i^n \int_0^t e^{-\gamma_i^n(t-s)}\mathrm{d}s$$

Deep Galerkin Method

$$\frac{\partial u}{\partial t} - \sum_{i,j=0}^{n} a^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{i=0}^{n} b^i \frac{\partial u}{\partial x_i} + ru = 0$$

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$$u(0, \mathbf{x}) = \Phi(\mathbf{x})$$

Minimize

$$\left\| \frac{\partial u}{\partial t} - \sum_{i,j=0}^{n} a^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{i=0}^{n} b^i \frac{\partial u}{\partial x_i} + ru \right\|_{[0,T]\times\Omega}^2 + \| u(0, \mathbf{x}) - \Phi(\mathbf{x}) \|_{\Omega}^2$$

Deep Galerkin Method

$$\frac{\partial u}{\partial t} - \sum_{i,j=0}^{n} a^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{i=0}^{n} b^i \frac{\partial u}{\partial x_i} + ru = 0$$

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Minimize

$$\left\| \frac{\partial u}{\partial t} - \sum_{i,j=0}^{n} a^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{i=0}^{n} b^i \frac{\partial u}{\partial x_i} + ru \right\|_{[0,T]\times\Omega}^2 + \| u(0, \mathbf{x}) - \Phi(\mathbf{x}) \|_{\Omega}^2$$

Issue: Taking second derivative makes training in high dimensions slow

Idea

Rewrite PDE as energy minimization problem

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- Only first order derivative
- No norm

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Split in symmetric and non-symmetric part

Splitting method

$$\frac{\partial u}{\partial t} = \sum_{i,j=0}^{n} a^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=0}^{n} b^i \frac{\partial u}{\partial x_i} - ru$$

Splitting method

$$\frac{\partial u}{\partial t} = \sum_{i,j=0}^{n} a^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=0}^{n} b^i \frac{\partial u}{\partial x_i} - ru$$

$$= \sum_{i,j=0}^{n} \frac{\partial}{\partial x_j}\left(a^{ij}\frac{\partial u}{\partial x_i}\right) - \sum_{i,j=0}^{n} \frac{\partial a^{ij}}{\partial x_j}\frac{\partial u}{\partial x_i} + \sum_{i=0}^{n} b^i \frac{\partial u}{\partial x_i} - ru$$

Splitting method

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \sum_{i,j=0}^{n} a^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=0}^{n} b^i \frac{\partial u}{\partial x_i} - ru \\
&= \sum_{i,j=0}^{n} \frac{\partial}{\partial x_j} \left(a^{ij} \frac{\partial u}{\partial x_i} \right) - \sum_{i,j=0}^{n} \frac{\partial a^{ij}}{\partial x_j} \frac{\partial u}{\partial x_i} + \sum_{i=0}^{n} b^i \frac{\partial u}{\partial x_i} - ru \\
&= \sum_{i,j=0}^{n} \frac{\partial}{\partial x_j} \left(a^{ij} \frac{\partial u}{\partial x_i} \right) - \sum_{i=0}^{n} \left(\sum_{j=0}^{n} \frac{\partial a^{ij}}{\partial x_j} - b^i \right) \frac{\partial u}{\partial x_i} - ru
\end{aligned}
$$

Splitting method

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\begin{aligned}
\frac{\partial u}{\partial t} &= \sum_{i,j=0}^{n} a^{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=0}^{n} b^i \frac{\partial u}{\partial x_i} - ru \\
&= \sum_{i,j=0}^{n} \frac{\partial}{\partial x_j}\left(a^{ij}\frac{\partial u}{\partial x_i}\right) - \sum_{i,j=0}^{n} \frac{\partial a^{ij}}{\partial x_j}\frac{\partial u}{\partial x_i} + \sum_{i=0}^{n} b^i \frac{\partial u}{\partial x_i} - ru \\
&= \sum_{i,j=0}^{n} \frac{\partial}{\partial x_j}\left(a^{ij}\frac{\partial u}{\partial x_i}\right) - \sum_{i=0}^{n}\left(\sum_{j=0}^{n}\frac{\partial a^{ij}}{\partial x_j} - b^i\right)\frac{\partial u}{\partial x_i} - ru \\
&= \nabla\cdot(A\nabla u) - ru - F(u) \\
F(u) &= \mathbf{b}\cdot\nabla u
\end{aligned}
$$

Example, $n = 1$

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sqrt{V_t}S_t\mathrm{d}W_t$$
$$\mathrm{d}V_t = -\lambda V_t\mathrm{d}t + \eta\sqrt{V_t}\mathrm{d}B_t$$

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$$\mathrm{d}V_t = -\lambda V_t\mathrm{d}t + \eta\sqrt{V_t}\mathrm{d}B_t$$

$$\frac{\partial u}{\partial t} = rS\frac{\partial u}{\partial S} - \lambda V\frac{\partial u}{\partial V} + \frac{1}{2}S^2V\frac{\partial^2 u}{\partial S^2} + \frac{1}{2}\eta^2V\frac{\partial^2 u}{\partial V^2} + \rho\eta SV\frac{\partial^2 u}{\partial S\partial V} - ru$$

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Example: $n = 1$

$$\begin{aligned}\frac{\partial u}{\partial t} =& rS\frac{\partial u}{\partial S} - \lambda V\frac{\partial u}{\partial V} + \frac{1}{2}S^2 V\frac{\partial^2 u}{\partial S^2} + \frac{1}{2}\eta^2 V\frac{\partial^2 u}{\partial V^2} + \rho\eta SV\frac{\partial^2 u}{\partial S\partial V} - ru \\ =& rS\frac{\partial u}{\partial S} - \lambda V\frac{\partial u}{\partial V} + \frac{\partial}{\partial S}\left(\frac{1}{2}S^2 V\frac{\partial u}{\partial S}\right) - SV\frac{\partial u}{\partial S}\end{aligned}$$

Example: $n = 1$

$$
\begin{aligned}
\frac{\partial u}{\partial t} =& rS\frac{\partial u}{\partial S} - \lambda V\frac{\partial u}{\partial V} + \frac{1}{2}S^2V\frac{\partial^2 u}{\partial S^2} + \frac{1}{2}\eta^2 V\frac{\partial^2 u}{\partial V^2} + \rho\eta SV\frac{\partial^2 u}{\partial S\partial V} - ru \\
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&+ \frac{\partial}{\partial V}\left(\frac{1}{2}\eta^2 V\frac{\partial u}{\partial V}\right) - \frac{1}{2}\eta^2\frac{\partial u}{\partial V}
\end{aligned}
$$

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\frac{\partial u}{\partial t} =& rS\frac{\partial u}{\partial S} - \lambda V\frac{\partial u}{\partial V} + \frac{1}{2}S^2V\frac{\partial^2 u}{\partial S^2} + \frac{1}{2}\eta^2V\frac{\partial^2 u}{\partial V^2} + \rho\eta SV\frac{\partial^2 u}{\partial S\partial V} - ru \\
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&+ \frac{\partial}{\partial V}\left(\frac{1}{2}\eta^2V\frac{\partial u}{\partial V}\right) - \frac{1}{2}\eta^2\frac{\partial u}{\partial V} + \frac{\partial}{\partial S}\left(\frac{1}{2}\rho\eta SV\frac{\partial u}{\partial V}\right) - \frac{1}{2}\rho\eta V\frac{\partial u}{\partial V} \\
&+ \frac{\partial}{\partial V}\left(\frac{1}{2}\rho\eta SV\frac{\partial u}{\partial S}\right) - \frac{1}{2}\rho\eta S\frac{\partial u}{\partial S} - ru
\end{aligned}
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\begin{aligned}
\frac{\partial u}{\partial t} =& rS\frac{\partial u}{\partial S} - \lambda V\frac{\partial u}{\partial V} + \frac{1}{2}S^2V\frac{\partial^2 u}{\partial S^2} + \frac{1}{2}\eta^2 V\frac{\partial^2 u}{\partial V^2} + \rho\eta SV\frac{\partial^2 u}{\partial S\partial V} - ru \\
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&+ \frac{\partial}{\partial V}\left(\frac{1}{2}\eta^2 V\frac{\partial u}{\partial V}\right) - \frac{1}{2}\eta^2\frac{\partial u}{\partial V} + \frac{\partial}{\partial S}\left(\frac{1}{2}\rho\eta SV\frac{\partial u}{\partial V}\right) - \frac{1}{2}\rho\eta V\frac{\partial u}{\partial V} \\
&+ \frac{\partial}{\partial V}\left(\frac{1}{2}\rho\eta SV\frac{\partial u}{\partial S}\right) - \frac{1}{2}\rho\eta S\frac{\partial u}{\partial S} - ru \\
=& \nabla\cdot\left(\frac{1}{2}\begin{bmatrix} S^2V & \rho\eta SV \\ \rho\eta SV & \eta^2 V\end{bmatrix}\nabla u\right) - \begin{bmatrix} SV + \frac{1}{2}\rho\eta S - rS \\ \frac{1}{2}\eta^2 + \frac{1}{2}\rho\eta V + \lambda V\end{bmatrix}\cdot\nabla u - ru
\end{aligned}
$$

Time Deep Gradient Flow

$$\begin{cases} u_t - \nabla \cdot (A \nabla u) + ru + F(u) = 0 & (t, \mathbf{x}) \in [0, T] \times \Omega \\ u(0, \mathbf{x}) = \Phi(\mathbf{x}) & \mathbf{x} \in \Omega \end{cases}$$

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- Divide $[0, T]$ in intervals $(t_{k-1}, t_k]$ with $h = t_k - t_{k-1}$

$$\frac{U^k - U^{k-1}}{h} - \nabla \cdot \left(A \nabla U^k \right) + rU^k + F\left(U^{k-1} \right) = 0$$

Time Deep **Gradient Flow**

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$$0 = \int_\Omega \left(\left(U^k - U^{k-1} \right) + h \left(-\nabla \cdot \left(A \nabla U^k \right) + rU^k + F\left(U^{k-1} \right) \right) \right) v \mathrm{d}\mathbf{x}$$

Time Deep **Gradient Flow**

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$$i^k(\tau) = I^k(U^k + \tau v)$$

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$$\frac{U^k - U^{k-1}}{h} - \nabla \cdot \left(A \nabla U^k\right) + rU^k + F(U^{k-1}) = 0$$

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$$i^k(\tau) = I^k(U^k + \tau v)$$

$$\begin{aligned} I^k(u) &= \frac{1}{2} \left\| u - U^{k-1} \right\|^2 + h \int_\Omega \frac{1}{2} \left((\nabla u)^T A \nabla u + ru^2 \right) + F\left(U^{k-1} \right) u dx \\ U^k &= \underset{u \in H^1(\Omega)}{\arg\min}\, I^k(u) \end{aligned}$$

Time **Deep** Gradient Flow

$$I^k(u) = \frac{1}{2}\left\| u - U^{k-1} \right\|^2 + h \int_\Omega \frac{1}{2}\left((\nabla u)^T A \nabla u + r u^2 \right) + F\left(U^{k-1}\right) u dx$$

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Time **Deep** Gradient Flow

$$
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I^k(u) &= \frac{1}{2}\left\|u - U^{k-1}\right\|^2 + h\int_\Omega \frac{1}{2}\left(\left(\nabla u\right)^T A \nabla u + ru^2\right) + F\left(U^{k-1}\right) u dx \\
U^k &= \underset{u \in H^1(\Omega)}{\arg\min}\, I^k(u) \\
f^k(\mathbf{x};\theta) &= \underset{u \in \mathcal{C}(\theta)}{\arg\min}\, I^k(u) \\
\mathcal{C}(\theta) &= \text{space of neural networks with parameters } \theta
\end{aligned}
$$

Algorithm

for each time step $k = 1, ..., K$ **do**
    Initialize $\theta_0^k = \theta^{k-1}$

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        Calculate the cost functional $I^k(f(\mathbf{x}^i; \theta_n^k))$

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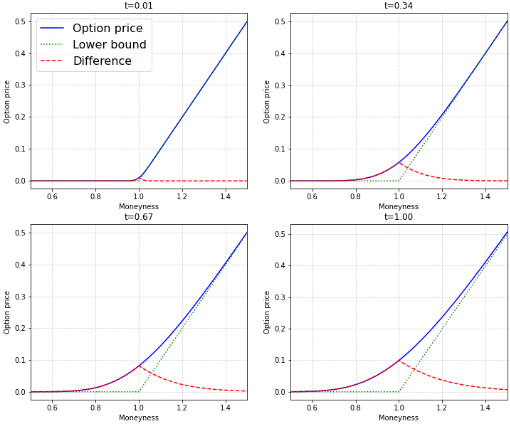
  Initialize $\theta_0^k = \theta^{k-1}$
  for each sampling stage n **do**
    Generate random points $\mathbf{x}^i$ for training
    Calculate the cost functional $I^k(f(\mathbf{x}^i; \theta_n^k))$
    Take a descent step $\theta_{n+1}^k = \theta_n^k - \alpha_n \nabla_\theta I^k(f(\mathbf{x}^i; \theta_n^k))$
  end for
end for

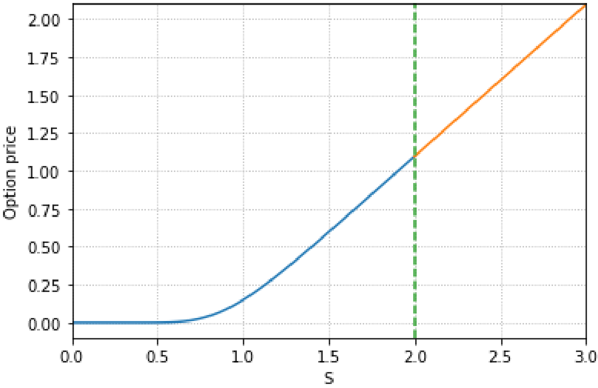
Base

No-arbitrage bound: $u(t, S) \geq S - Ke^{-rt}$



Linearization

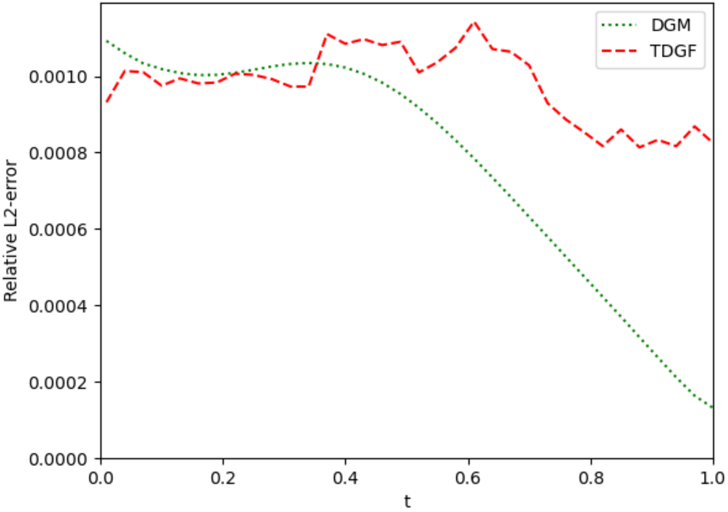
$$f(x_p + y; \theta) = f(x_p; \theta) + y, \quad y > 0.$$



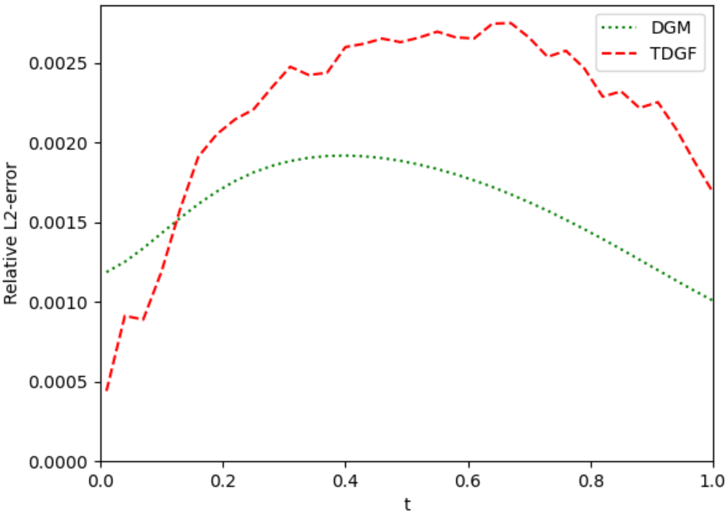
Architecture

$$
\begin{aligned}
S^1 &= \sigma_1\left(W^1\mathbf{x} + b^1\right) && \\
Z^l &= \sigma_1\left(U^{z,l}\mathbf{x} + W^{z,l}S^l + b^{z,l}\right) && l = 1, ..., L \\
G^l &= \sigma_1\left(U^{g,l}\mathbf{x} + W^{g,l}S^l + b^{g,l}\right) && l = 1, ..., L \\
R^l &= \sigma_1\left(U^{r,l}\mathbf{x} + W^{r,l}S^l + b^{r,l}\right) && l = 1, ..., L \\
H^l &= \sigma_1\left(U^{h,l}\mathbf{x} + W^{h,l}\left(S^l \odot R^l\right) + b^{h,l}\right) && l = 1, ..., L \\
S^{l+1} &= \left(1 - G^l\right) \odot H^l + Z^l \odot S^l && l = 1, ..., L \\
f(\mathbf{x};\theta) &= \text{base} + \sigma_2\left(WS^{L+1} + b\right) && \sigma_2 > 0
\end{aligned}
$$

Lifted Heston, $n = 1$



Lifted Heston, $n = 20$



Running times

Model	LH, n=1	LH, n=20
DGM	12.5×10^3	54.3×10^3
TDGF	5.9×10^3	7.4×10^3

Table: Training time

Running times

Model	LH, n=1	LH, n=20
DGM	12.5×10^3	54.3×10^3
TDGF	5.9×10^3	7.4×10^3

Table: Training time

Model	LH, n=1	LH, n=20
COS	9.1	10.0
DGM	0.0043	0.0015
TDGF	0.0019	0.0018

Table: Computing time

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