Time Deep Gradient Flow Method for Option Pricing

Winter school on Mathematical Finance

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joint work with Chenguang Liu & Antonis Papapantoleon

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Option Pricing

$$\frac{\partial u}{\partial t} - \sum_{i,j=0}^{n} a^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{i=0}^{n} b^i \frac{\partial u}{\partial x_i} + ru = 0,$$

$$u(0, x) = \Phi(x)$$

Deep Galerkin Method

$$\frac{\partial u}{\partial t} - \sum_{i,j=0}^{n} a^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{i=0}^{n} b^i \frac{\partial u}{\partial x_i} + ru = 0$$

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Minimize

$$\left\| \frac{\partial u}{\partial t} - \sum_{i,j=0}^{n} a^{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{i=0}^{n} b^i \frac{\partial u}{\partial x_i} + ru \right\|^2_{[0,T]\times\Omega} + \|u(0,x) - \Phi(x)\|^2_\Omega$$

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Issue: Taking second derivative makes training in high dimensions slow

Idea

Rewrite PDE as energy minimization problem

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- Only first order derivative
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Split in symmetric and non-symmetric part

Splitting method

$$\frac{\partial u}{\partial t} = \sum_{i,j=0}^{n} a^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=0}^{n} b^i \frac{\partial u}{\partial x_i} - ru$$

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$$= \sum_{i,j=0}^{n} \frac{\partial}{\partial x_j} \left(a^{ij} \frac{\partial u}{\partial x_i} \right) - \sum_{i,j=0}^{n} \frac{\partial a^{ij}}{\partial x_j} \frac{\partial u}{\partial x_i} + \sum_{i=0}^{n} b^i \frac{\partial u}{\partial x_i} - ru$$

Splitting method

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\begin{aligned}
\frac{\partial u}{\partial t} &= \sum_{i,j=0}^{n} a^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=0}^{n} b^i \frac{\partial u}{\partial x_i} - ru \\
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$$= \nabla \cdot (A\nabla u) - ru - F(u)$$

$$F(u) = \mathbf{b} \cdot \nabla u$$

Example: Heston model

$$
\begin{aligned}
dS_t &= rS_t dt + \sqrt{V_t} S_t dW_t & S_0 > 0 \\
dV_t &= \kappa(\theta - V_t)dt + \eta\sqrt{V_t} dB_t & V_0 > 0
\end{aligned}
$$

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$$\frac{\partial u}{\partial t} = rS\frac{\partial u}{\partial S} + \kappa(\theta - V)\frac{\partial u}{\partial V} + \frac{1}{2}S^2 V\frac{\partial^2 u}{\partial S^2} + \frac{1}{2}\eta^2 V\frac{\partial^2 u}{\partial V^2} + \rho\eta SV\frac{\partial^2 u}{\partial S\partial V} - ru$$

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& + \frac{\partial}{\partial V}\left(\frac{1}{2}\eta^2V\frac{\partial u}{\partial V}\right) - \frac{1}{2}\eta^2\frac{\partial u}{\partial V} + \frac{\partial}{\partial S}\left(\frac{1}{2}\rho\eta SV\frac{\partial u}{\partial V}\right) - \frac{1}{2}\rho\eta V\frac{\partial u}{\partial V} \\
& + \frac{\partial}{\partial V}\left(\frac{1}{2}\rho\eta SV\frac{\partial u}{\partial S}\right) - \frac{1}{2}\rho\eta S\frac{\partial u}{\partial S} - ru \\
=& \nabla \cdot \left(\frac{1}{2}\begin{bmatrix} S^2V & \rho\eta SV \\ \rho\eta SV & \eta^2V \end{bmatrix}\nabla u\right) - ru \\
& - \begin{bmatrix} SV - rS + \frac{1}{2}\rho\eta S \\ \kappa(V - \theta) + \frac{1}{2}\rho\eta V + \frac{1}{2}\eta^2 \end{bmatrix} \cdot \nabla u
\end{aligned}
$$

Time Deep Gradient Flow

$$\begin{cases} \frac{\partial u}{\partial t} - \nabla \cdot (A \nabla u) + ru + F(u) = 0 & (t, \mathbf{x}) \in [0, T] \times \Omega \\ u(0, \mathbf{x}) = \Phi(\mathbf{x}) & \mathbf{x} \in \Omega \end{cases}$$

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- Divide $[0, T]$ in intervals $(t_{k-1}, t_k]$ with $h = t_k - t_{k-1}$

$$\frac{U^k - U^{k-1}}{h} - \nabla \cdot \left(A \nabla U^k \right) + rU^k + F\left(U^{k-1} \right) = 0$$

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Theorem (Akrivis and Crouzeix 2004)

There exists a constant C independent of h and k such that

$$\max_{0 \le k \le N} \left\| u(t_k) - U^k \right\| \le Ch$$

Time Deep **Gradient Flow**

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Theorem

The minimizer $w_* \in \mathcal{H}_0^1\left(\mathbb{R}^d\right)$ of I^k is the unique solution U^k.

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$$i^k(\tau) = I^k(w_* + \tau v)$$

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$$0 = \left(i^k\right)'(0) \overset{IBP}{=} \int_{\mathbb{R}^d} \left(\left(w_* - U^{k-1}\right) + h\left(-\nabla \cdot (A\nabla w_*) + rw_* + F\left(U^{k-1}\right)\right)\right) v dx.$$

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Convergence of the minimizer

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Let w_m be a sequence in $\mathcal{H}_0^1\left(\mathbb{R}^d\right)$ and w_* the minimizer of I^k.

$$\lim_{m\to\infty} \|w_m - w_*\|_{\mathcal{H}_0^1} = 0 \quad \Longleftrightarrow \quad \lim_{m\to\infty} I^k\left(w_m\right) = I^k\left(w_*\right)$$

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$$\mathcal{L}^k(u) = \frac{1}{2}\|u\|^2 + \frac{h}{2}\int_{\mathbb{R}^d} (\nabla u)^T A\nabla u + ru^2\mathrm{d}x,$$

$$\mathcal{G}^k(u) = -\left\langle u, U^{k-1}\right\rangle + \frac{1}{2}\left\|U^{k-1}\right\|^2 + h\int_{\mathbb{R}^d} F\left(U^{k-1}\right) u\mathrm{d}x$$

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Since $I^k\left(w_m\right) \to I^k\left(w_*\right)$, $\mathcal{L}^k\left(w_m\right) \to \mathcal{L}^k\left(w_*\right)$.

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$$\frac{1+hr}{2}\left\|w_m - w_*\right\|^2 + \frac{h}{2}\left\|\sqrt{A}\nabla\left(w_m - w_*\right)\right\|^2 \to 0.$$

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$$\|w_m - w_*\|_{\mathcal{H}_0^1} \to 0. \qquad \square$$

Time **Deep** Gradient Flow

Definition (Activation function)

An activation function is a function $\psi : \mathbb{R}^d \to \mathbb{R}$ such that $\psi \in C_c^\infty\left(\mathbb{R}^d\right)$ and $\int_{\mathbb{R}^d} \psi(x) \,\mathrm{d}x \neq 0$.

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Definition (Neural network)

$$\mathcal{C}^N(\psi) = \left\{ f\left(\theta; x\right) : \mathbb{R}^d \to \mathbb{R} : f\left(\theta; x\right) = \sum_{i=1}^{N} \beta^i \psi\left(\alpha^i x + c^i\right) \right\},$$

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$$\mathcal{C}\left(\psi\right) = \cup_{N \geq 1} \mathcal{C}^N\left(\psi\right)$$

Theorem

$\mathcal{C}(\psi)$ is dense in $\mathcal{H}_0^1\left(\mathbb{R}^d\right)$.

Algorithm

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for each time step $k = 1, ..., N_t$ **do**
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    for each sampling stage n **do**
        Generate random points $\mathbf{x}^i$ for training.

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        Generate random points $\mathbf{x}^i$ for training.
        Calculate the cost functional $I^k(f(\theta_n^k; \mathbf{x}^i))$.
        Take a descent step $\theta_{n+1}^k = \theta_n^k - \eta_n \nabla_\theta I^k(f(\theta_n^k; \mathbf{x}^i))$.
    end for
end for

Convergence when training

Neural network:

$$V_t^N\left(\theta^N;x\right) = V^N\left(\theta_t^N;x\right) = N^{-\delta}\sum_{i=1}^{N}\beta^i\psi\left(\alpha^i x + c^i\right),$$

$\theta^N = \left(\beta^i, \alpha^i, c^i\right)_{i=1}^N$, $\frac{1}{2} < \delta < 1$.

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$$V_t^N \overset{N\to\infty}{\longrightarrow} V_t \overset{t\to\infty}{\longrightarrow} w_*$$

Gradient Descent

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$$\frac{\mathrm{d}\theta_t^N}{\mathrm{d}t} = -\eta_N \nabla_\theta I^k\left(V^N\left(\theta_t^N; x\right)\right)$$

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$\theta^N = \left(\beta^i, \alpha^i, c^i\right)_{i=1}^N$, $\frac{1}{2} < \delta < 1$. $\eta_N = N^{2\delta-1}$

$$\frac{\mathrm{d}\theta_t^N}{\mathrm{d}t} = -\eta_N \nabla_\theta I^k\left(V^N\left(\theta_t^N; x\right)\right)$$

$$\frac{\mathrm{d}V_t^N(x)}{\mathrm{d}t} = \nabla_\theta V^N\left(\theta_t^N; x\right)\cdot\frac{\mathrm{d}\theta_t^N}{\mathrm{d}t}$$

Gradient Descent

$$V^N\left(\theta_t^N;x\right)=N^{-\delta}\sum_{i=1}^{N}\beta^i\psi\left(\alpha^i x+c^i\right),$$

$\theta^N=\left(\beta^i,\alpha^i,c^i\right)_{i=1}^N$, $\frac{1}{2}<\delta<1$. $\eta_N=N^{2\delta-1}$

$$\frac{\mathrm{d}\theta_t^N}{\mathrm{d}t}=-\eta_N\nabla_\theta I^k\left(V^N\left(\theta_t^N;x\right)\right)$$

$$\begin{aligned}\frac{\mathrm{d}V_t^N(x)}{\mathrm{d}t}=&\nabla_\theta V^N\left(\theta_t^N;x\right)\cdot\frac{\mathrm{d}\theta_t^N}{\mathrm{d}t}\\=&-\eta_N\nabla_\theta V^N\left(\theta_t^N;x\right)\cdot\nabla_\theta I^k\left(V^N\left(\theta_t^N;x\right)\right)\end{aligned}$$

Wide network limit

$$\frac{\mathrm{d}V_t^N(x)}{\mathrm{d}t} = -\,\eta_N \nabla_\theta V^N\left(\theta_t^N; x\right) \cdot \nabla_\theta I^k\left(V^N\left(\theta_t^N; x\right)\right)$$

Wide network limit

$$\begin{aligned}\frac{\mathrm{d}V_t^N(x)}{\mathrm{d}t} &= -\eta_N \nabla_\theta V^N\left(\theta_t^N; x\right) \cdot \nabla_\theta I^k\left(V^N\left(\theta_t^N; x\right)\right) \\ &= -\left\langle \mathcal{D}I^k\left(V_t^N\right), Z_t^N(x, .)\right\rangle_{\mathcal{H}_0^1}\end{aligned}$$

$$Z_t^N(x, y) = N^{-1}\sum_{i=1}^{N} \nabla_{\beta,\alpha,c}\beta_t^i\psi\left(\alpha_t^i x + c_t^i\right) \cdot \nabla_{\beta,\alpha,c}\beta_t^i\psi\left(\alpha_t^i y + c_t^i\right)$$

Wide network limit

$$
\begin{aligned}
\frac{\mathrm{d} V_t^N(x)}{\mathrm{d} t} &= -\eta_N \nabla_\theta V^N\left(\theta_t^N; x\right) \cdot \nabla_\theta I^k\left(V^N\left(\theta_t^N; x\right)\right) \\
&= -\left\langle \mathcal{D} I^k\left(V_t^N\right), Z_t^N(x, .)\right\rangle_{\mathcal{H}_0^1} \\
\frac{\mathrm{d} V_t(x)}{\mathrm{d} t} &= -\left\langle \mathcal{D} I^k\left(V_t\right), Z(x, .)\right\rangle_{\mathcal{H}_0^1}
\end{aligned}
$$

$$
\begin{aligned}
Z_t^N(x, y) =& N^{-1} \sum_{i=1}^N \nabla_{\beta,\alpha,c} \beta_t^i \psi\left(\alpha_t^i x + c_t^i\right) \cdot \nabla_{\beta,\alpha,c} \beta_t^i \psi\left(\alpha_t^i y + c_t^i\right) \\
Z(x, y) =& \mathbb{E}\left[\nabla_{\beta,\alpha,c} \beta_0^1 \psi\left(\alpha_0^1 x + c_0^1\right) \cdot \nabla_{\beta,\alpha,c} \beta_0^1 \psi\left(\alpha_0^1 y + c_0^1\right)\right]
\end{aligned}
$$

Wide network limit

$$\frac{\mathrm{d}V_t^N(x)}{\mathrm{d}t} = -\left\langle \mathcal{D}I^k\left(V_t^N\right), Z_t^N(x,.)\right\rangle_{\mathcal{H}_0^1}$$

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$$Z(x,y) = \mathbb{E}\left[\nabla_{\beta,\alpha,c}\beta_0^1\psi\left(\alpha_0^1x + c_0^1\right)\cdot\nabla_{\beta,\alpha,c}\beta_0^1\psi\left(\alpha_0^1y + c_0^1\right)\right]$$

Wide network limit

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Theorem

For any $T > 0$,

$$\sup_{0\le t\le T}\mathbb{E}\left[\left\|V_t^N - V_t\right\|_{\mathcal{H}_0^1}\right] \overset{N\to\infty}{\longrightarrow} 0.$$

Convergence in time

Theorem

$$\lim_{t\to\infty} \|V_t - w_*\|_{\mathcal{H}_0^1} = 0.$$

$$\frac{\mathrm{d}V_t(x)}{\mathrm{d}t} = -\left\langle \mathcal{D}I^k(V_t), Z(x,.)\right\rangle_{\mathcal{H}_0^1}$$

Convergence in time

Theorem

$$\lim_{t\to\infty} \|V_t - w_*\|_{\mathcal{H}_0^1} = 0.$$

$$\frac{\mathrm{d}V_t(x)}{\mathrm{d}t} = -\left\langle \mathcal{D}I^k(V_t), Z(x,.)\right\rangle_{\mathcal{H}_0^1}$$

$$\frac{\mathrm{d}(V_t - w_*)(x)}{\mathrm{d}t} = -\left\langle \mathcal{D}I^k(V_t - w_* + w_*), Z(x,.)\right\rangle_{\mathcal{H}_0^1}$$

$$= -\tilde{\mathcal{T}}(V_t - w_*)(x)$$

Convergence in time

Proof: $\lim_{t\to\infty} \|V_t - w_*\|_{\mathcal{H}_0^1} = 0$.

$\tilde{\mathcal{T}}$ is a self-adjoint, positive definite trace class operator. Spectral decomposition:

$$\tilde{\mathcal{T}}(\tilde{e}_i) = \lambda_i \tilde{e}_i,$$

$\lambda_1 \geq \lambda_2 \geq ... > 0$, orthogonal basis $\{\tilde{e}_i\}_{i=1}^{\infty}$.

Convergence in time

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$$\begin{aligned}\frac{\mathrm{d}h_t^i}{\mathrm{d}t} :=& \frac{\langle \mathrm{d}(V_t - w_*), \tilde{e}_i \rangle}{\mathrm{d}t} = -\left\langle \tilde{\mathcal{T}}(V_t - w_*), \tilde{e}_i \right\rangle = -\left\langle V_t - w_*, \tilde{\mathcal{T}}(\tilde{e}_i) \right\rangle \\ =& -\lambda_i h_t^i.\end{aligned}$$

Convergence in time

Proof: $\lim_{t\to\infty} \|V_t - w_*\|_{\mathcal{H}_0^1} = 0$.

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$h_t^i = e^{-\lambda_i t} h_0^i$.

Convergence in time

Proof: $\lim_{t\to\infty} \| V_t - w_* \|_{\mathcal{H}_0^1} = 0$.

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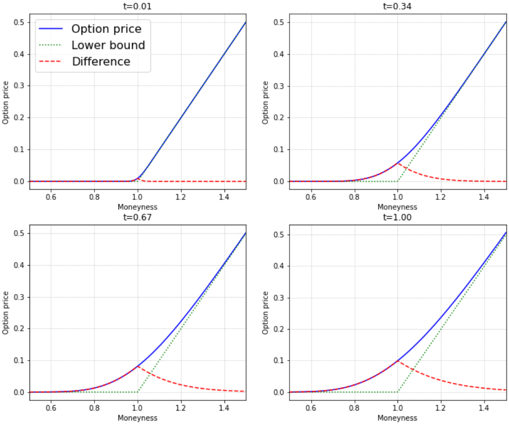
$h_t^i = e^{-\lambda_i t} h_0^i$. Parseval's identity:

$$\| V_t - w_* \|^2 = \sum_{i=1}^{\infty} \left(h_t^i\right)^2 = \sum_{i=1}^{\infty} e^{-2\lambda_i t} \left(h_0^i\right)^2 \xrightarrow{t\to\infty} 0. \qquad \square$$

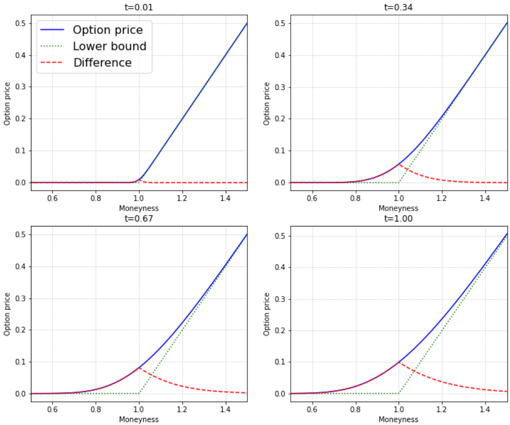
Architecture: no-arbitrage bound

European call: $u(t, S) \geq S - Ke^{-rt}$



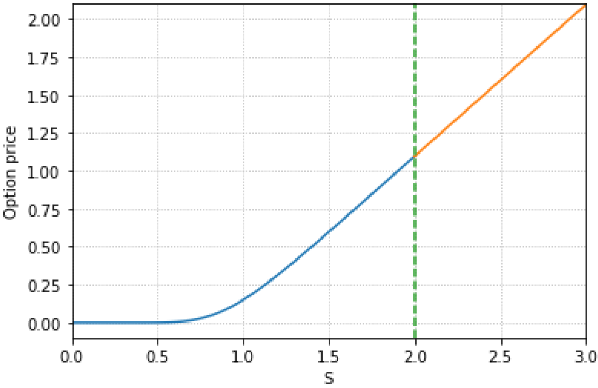
Architecture: no-arbitrage bound

$$\text{European call:} \quad u(t,S) \geq S - Ke^{-rt}$$
$$\text{American put:} \quad u(t,S) \geq K - S$$

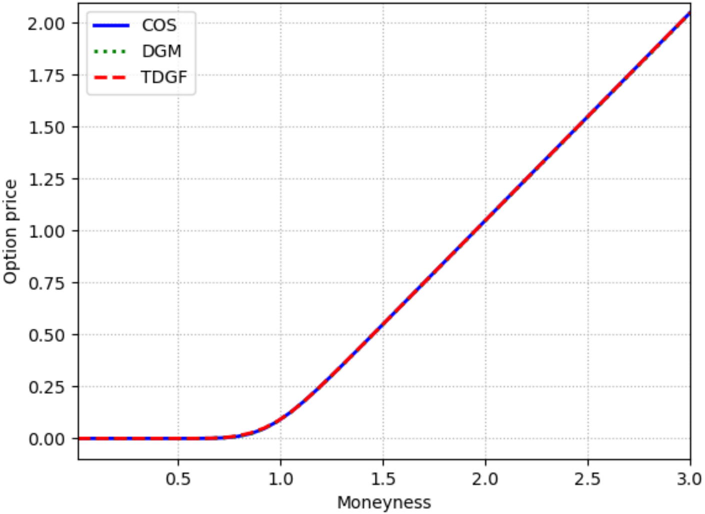


Architecture: linearization

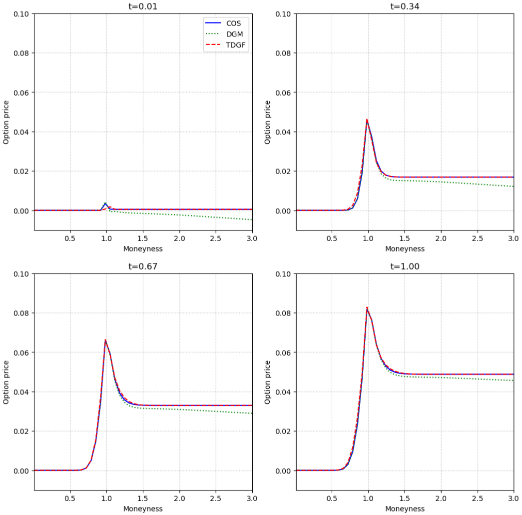
$$u(x_p + y; \theta) = u(x_p; \theta) + y, \quad y > 0.$$



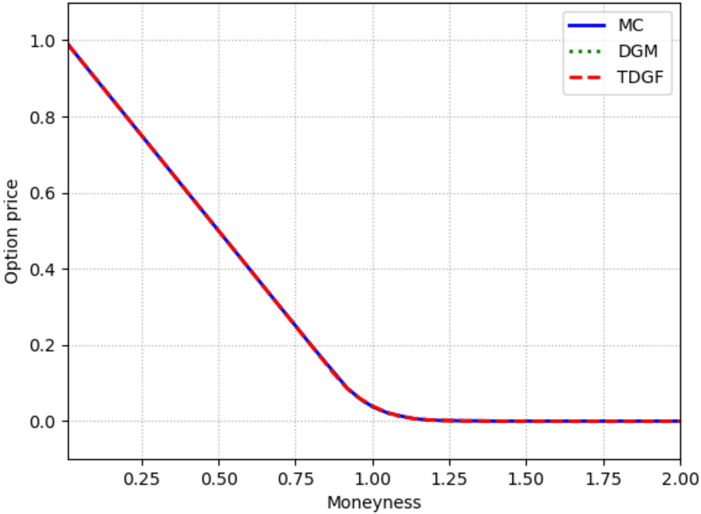
European Option, $d = 1$



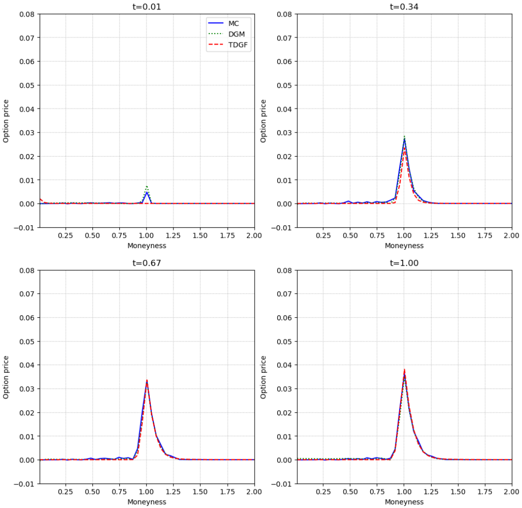
European Option, $d = 1$



American Option, $d = 5$



American Option, $d = 5$



Running times

Model	European, $d = 1$	American, $d = 5$
DGM	12.5×10^3	42.1×10^3
TDGF	6.0×10^3	12.9×10^3

Table: Training time

Running times

Model	European, $d = 1$	American, $d = 5$
DGM	12.5×10^3	42.1×10^3
TDGF	6.0×10^3	12.9×10^3

Table: Training time

Model	European, $d = 1$	American, $d = 5$
COS/MC	0.018	5.82
DGM	0.0016	0.0018
TDGF	0.0060	0.0076

Table: Computing time

Time Deep Gradient Flow Method for Option Pricing

Winter school on Mathematical Finance

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January 21, 2025

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