Time-Stepping Deep Gradient Flow Methods

Seminar National Technical University of Athens

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joint work with Chenguang Liu & Antonis Papapantoleon

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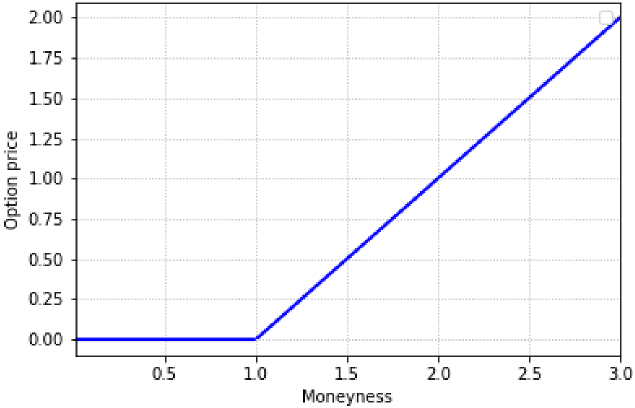
Options

A contract which gives the owner the right, but not the obligation, to buy a stock at a price K at a future time T

Pay-off

$$\Phi(S_T) = (S_T - K)^+$$



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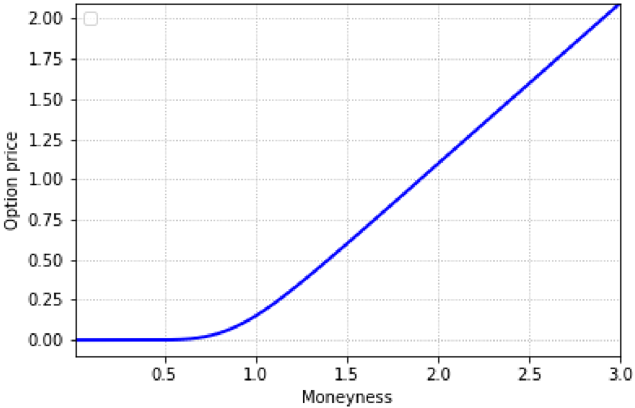

Stock price

$$dS_t = rS_t dt + \sqrt{V_t} S_t dW_t \qquad S_0 > 0$$
$$dV_t = \kappa(\theta - V_t)dt + \eta\sqrt{V_t} dB_t \qquad V_0 > 0$$

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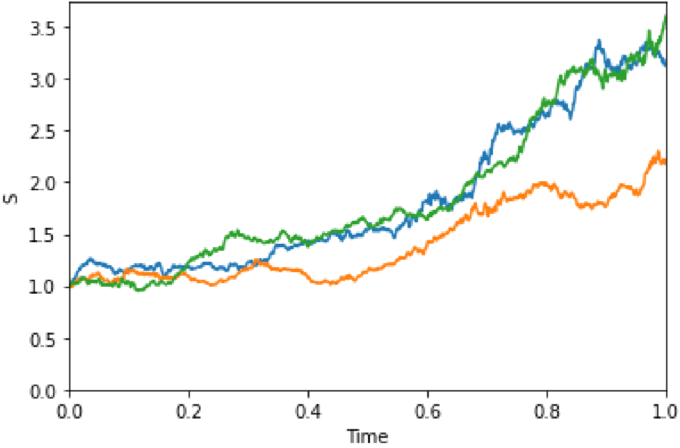

Pricing

Price of a derivative with pay-off $\Phi(S_T)$

$$u(t, S) = \mathbb{E}\left[\Phi(S_T) | S_t\right]$$

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$$\frac{\partial u}{\partial t} + \sum_{i,j=0}^{n} a^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=0}^{n} b^i \frac{\partial u}{\partial x_i} - ru = 0,$$

$$u(T, x) = \Phi(x)$$

Pricing

Price of a derivative with pay-off $\Phi(S_T)$

$$u(t) = \mathbb{E}\left[e^{-r(T-t)}\Phi(S_T)|S_t\right]$$

$$\frac{\partial u}{\partial t} - \sum_{i,j=0}^{n} a^{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{i=0}^{n} b^i \frac{\partial u}{\partial x_i} + ru = 0,$$

$$u(0,x) = \Phi(x)$$

Deep Galerkin Method

$$\frac{\partial u}{\partial t} - \sum_{i,j=0}^{n} a^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{i=0}^{n} b^i \frac{\partial u}{\partial x_i} + ru = 0$$

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Minimize

$$\left\| \frac{\partial u}{\partial t} - \sum_{i,j=0}^{n} a^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{i=0}^{n} b^i \frac{\partial u}{\partial x_i} + ru \right\|_{[0,T]\times\Omega}^2 + \|u(0,x) - \Phi(x)\|_{\Omega}^2$$

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Issue: Taking second derivative makes training in high dimensions slow

Idea

Rewrite PDE as energy minimization problem

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- Only first order derivative
- No norm

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Split in symmetric and non-symmetric part

Splitting method

$$\frac{\partial u}{\partial t} = \sum_{i,j=0}^{n} a^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=0}^{n} b^i \frac{\partial u}{\partial x_i} - ru$$

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$$\frac{\partial u}{\partial t} = \sum_{i,j=0}^{n} a^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=0}^{n} b^i \frac{\partial u}{\partial x_i} - ru$$

$$= \sum_{i,j=0}^{n} \frac{\partial}{\partial x_j} \left(a^{ij} \frac{\partial u}{\partial x_i} \right) - \sum_{i,j=0}^{n} \frac{\partial a^{ij}}{\partial x_j} \frac{\partial u}{\partial x_i} + \sum_{i=0}^{n} b^i \frac{\partial u}{\partial x_i} - ru$$

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\begin{aligned}
\frac{\partial u}{\partial t} &= \sum_{i,j=0}^{n} a^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=0}^{n} b^i \frac{\partial u}{\partial x_i} - ru \\
&= \sum_{i,j=0}^{n} \frac{\partial}{\partial x_j}\left(a^{ij}\frac{\partial u}{\partial x_i}\right) - \sum_{i,j=0}^{n} \frac{\partial a^{ij}}{\partial x_j}\frac{\partial u}{\partial x_i} + \sum_{i=0}^{n} b^i \frac{\partial u}{\partial x_i} - ru \\
&= \sum_{i,j=0}^{n} \frac{\partial}{\partial x_j}\left(a^{ij}\frac{\partial u}{\partial x_i}\right) - \sum_{i=0}^{n} \left(\sum_{j=0}^{n} \frac{\partial a^{ij}}{\partial x_j} - b^i\right) \frac{\partial u}{\partial x_i} - ru
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&= \nabla \cdot (A \nabla u) - ru - F(u) \\
F(u) &= \mathbf{b} \cdot \nabla u
\end{aligned}
$$

Example: Heston model

$$
\begin{aligned}
dS_t &= rS_t dt + \sqrt{V_t} S_t dW_t & \qquad S_0 > 0 \\
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\frac{\partial u}{\partial t} = rS \frac{\partial u}{\partial S} + \kappa(\theta - V) \frac{\partial u}{\partial V} + \frac{1}{2} S^2 V \frac{\partial^2 u}{\partial S^2} + \frac{1}{2} \eta^2 V \frac{\partial^2 u}{\partial V^2} + \rho \eta S V \frac{\partial^2 u}{\partial S \partial V} - ru
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=& rS\frac{\partial u}{\partial S} + \kappa(\theta - V)\frac{\partial u}{\partial V} + \frac{\partial}{\partial S}\left(\frac{1}{2}S^2 V\frac{\partial u}{\partial S}\right) - SV\frac{\partial u}{\partial S} \\
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& + \frac{\partial}{\partial V}\left(\frac{1}{2}\rho\eta SV\frac{\partial u}{\partial S}\right) - \frac{1}{2}\rho\eta S\frac{\partial u}{\partial S} - ru \\
=& \nabla \cdot \left(\frac{1}{2}\begin{bmatrix} S^2V & \rho\eta SV \\ \rho\eta SV & \eta^2 V \end{bmatrix}\nabla u\right) - ru \\
& - \begin{bmatrix} SV - rS + \frac{1}{2}\rho\eta S \\ \kappa(V-\theta) + \frac{1}{2}\rho\eta V + \frac{1}{2}\eta^2 \end{bmatrix} \cdot \nabla u
\end{aligned}
$$

Time Deep Gradient Flow

$$\begin{cases} u_t - \nabla \cdot (A \nabla u) + ru + F(u) = 0 & (t, \mathbf{x}) \in [0, T] \times \Omega \\ u(0, \mathbf{x}) = \Phi(\mathbf{x}) & \mathbf{x} \in \Omega \end{cases}$$

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- Divide $[0, T]$ in intervals $(t_{k-1}, t_k]$ with $h = t_k - t_{k-1}$

$$\frac{U^k - U^{k-1}}{h} - \nabla \cdot \left(A \nabla U^k \right) + rU^k + F\left(U^{k-1} \right) = 0$$
$$U^0 = \Phi$$

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Theorem (Akrivis and Crouzeix 2004)

There exists a constant C independent of h and k such that

$$\max_{0 \leq k \leq N} \left\| u(t_k) - U^k \right\| \leq Ch$$

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Theorem

The minimizer $w_* \in \mathcal{H}_0^1\left(\mathbb{R}^d\right)$ of I^k is the unique solution U^k.

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$$\begin{aligned} 0 &= \left(i^k\right)'(0) \\ &= \int_{\mathbb{R}^d} \left(\left(w_* - U^{k-1}\right) + h\left(-\nabla \cdot (A\nabla w_*) + r w_* + F\left(U^{k-1}\right)\right)\right) v dx. \end{aligned}$$

$\square$

Time **Deep** Gradient Flow

Definition (Activation function)

An activation function is a function $\psi : \mathbb{R}^d \to \mathbb{R}$ such that $\psi \in C_c^\infty\left(\mathbb{R}^d\right)$ and $\int_{\mathbb{R}^d} \psi\left(x\right) \mathrm{d}x \neq 0$.

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Definition (Neural network)

$$\mathcal{C}^n(\psi) = \left\{ \zeta\left(x\right) : \mathbb{R}^d \to \mathbb{R} : \zeta\left(x\right) = \sum_{i=1}^{n} \beta_i \psi\left(\alpha_i x + c_i\right) \right\},$$

$$\mathcal{C}\left(\psi\right) = \cup_{n \geq 1} \mathcal{C}^n\left(\psi\right)$$

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Theorem

$\mathcal{C}(\psi)$ is dense in $\mathcal{H}_0^1\left(\mathbb{R}^d\right)$.

Convergence of the minimizer

Theorem

Let w_m be a sequence in $\mathcal{H}_0^1\left(\mathbb{R}^d\right)$ and w_* the minimizer of I^k.

$$\lim_{m\to\infty} \|w_m - w_*\|_{\mathcal{H}_0^1} = 0 \quad \Longleftrightarrow \quad \lim_{m\to\infty} I^k\left(w_m\right) = I^k\left(w_*\right)$$

$$I^k\left(u\right) = \frac{1}{2}\left\|u - U^{k-1}\right\|^2 + h\int_{\mathbb{R}^d} \frac{1}{2}\left(\left(\nabla u\right)^T A\nabla u + ru^2\right) + F\left(U^{k-1}\right) u\mathrm{d}x$$

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&=: \mathcal{L}^k(u) + \mathcal{G}^k(u),\\
\mathcal{L}^k(u) &= \frac{1}{2}\|u\|^2 + \frac{h}{2}\int_{\mathbb{R}^d} (\nabla u)^T A\nabla u + ru^2\mathrm{d}x,\\
\mathcal{G}^k(u) &= -\left\langle u, U^{k-1}\right\rangle + \frac{1}{2}\left\|U^{k-1}\right\|^2 + h\int_{\mathbb{R}^d} F\left(U^{k-1}\right) u\mathrm{d}x
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$\Longrightarrow$ $\quad I^k$ is continuous.

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Since $I^k\left(w_m\right) \to I^k\left(w_*\right)$, $\mathcal{L}^k\left(w_m\right) \to \mathcal{L}^k\left(w_*\right)$.

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$$\frac{1+hr}{2}\left\|w_m - w_*\right\|^2 + \frac{h}{2}\left\|\sqrt{A}\nabla\left(w_m - w_*\right)\right\|^2 \to 0.$$

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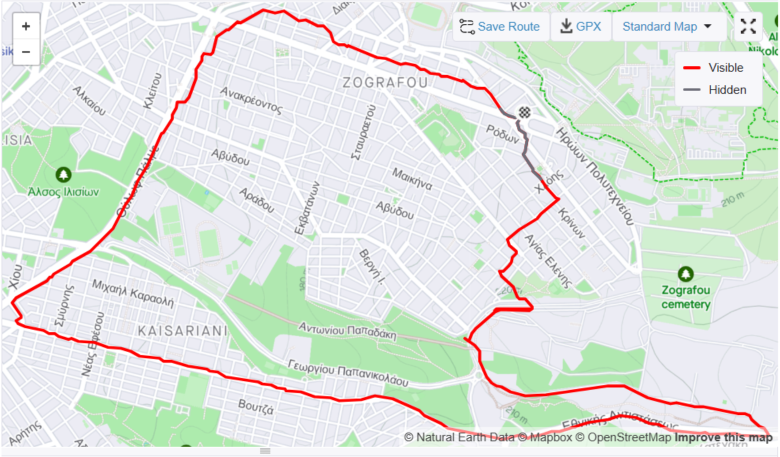
$\left\|w_m - w_*\right\|_{\mathcal{H}_0^1} \to 0.$ $\square$

Intermezzo: Gradient Descent



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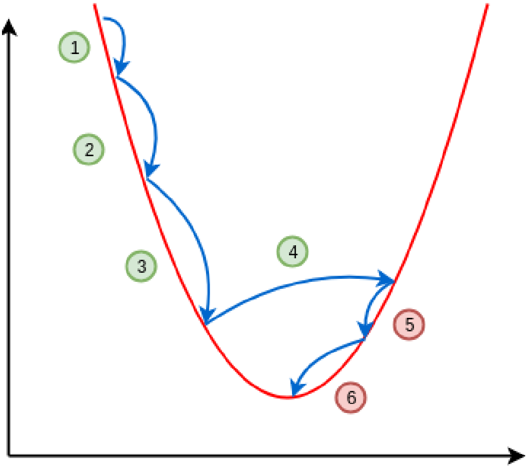

Intermezzo: Gradient Descent

$$\theta_{new} = \theta - \eta \nabla f(\theta)$$

Convergence when training

Neural network:

$$V_t^N\left(\theta^N; x\right) = V^N\left(\theta_t^N; x\right) = N^{-\delta}\sum_{i=1}^{N}\beta^i\psi\left(\alpha^i x + c^i\right),$$

$\theta^N = \left(\beta^i, \alpha^i, c^i\right)_{i=1}^N$, $\frac{1}{2} < \delta < 1$.

Convergence when training

Neural network:

$$V_t^N\left(\theta^N;x\right) = V^N\left(\theta_t^N;x\right) = N^{-\delta}\sum_{i=1}^{N}\beta^i\psi\left(\alpha^i x + c^i\right),$$

$\theta^N = \left(\beta^i, \alpha^i, c^i\right)_{i=1}^N$, $\frac{1}{2} < \delta < 1$.

$$V_t^N \overset{N\to\infty}{\longrightarrow} V_t \overset{t\to\infty}{\longrightarrow} w_*$$

Gradient Descent

$$V^N\left(\theta^N; x\right) = N^{-\delta} \sum_{i=1}^{N} \beta^i \psi\left(\alpha^i x + c^i\right),$$

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$$\frac{\mathrm{d}\theta_t^N}{\mathrm{d}t} = -\eta_N \nabla_\theta I^k\left(V^N\left(\theta_t^N; x\right)\right)$$

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$$\begin{aligned}\frac{\mathrm{d}V_t^N(x)}{\mathrm{d}t} =& \nabla_\theta V^N\left(\theta_t^N; x\right)\cdot\frac{\mathrm{d}\theta_t^N}{\mathrm{d}t}\\ =& -\eta_N\nabla_\theta V^N\left(\theta_t^N; x\right)\cdot\nabla_\theta I^k\left(V^N\left(\theta_t^N; x\right)\right)\end{aligned}$$

Wide network limit

$$\frac{\mathrm{d}V_t^N(x)}{\mathrm{d}t} = -\eta_N \nabla_\theta V^N\left(\theta_t^N; x\right) \cdot \nabla_\theta I^k\left(V^N\left(\theta_t^N; x\right)\right)$$

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$$Z_t^N(x, y) = N^{-1}\sum_{i=1}^{N} \nabla_{\beta,\alpha,c}\beta_t^i\psi\left(\alpha_t^i x + c_t^i\right) \cdot \nabla_{\beta,\alpha,c}\beta_t^i\psi\left(\alpha_t^i y + c_t^i\right)$$

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$$= -\left\langle \mathcal{D}I^k\left(V_t^N\right), Z_t^N(x, .)\right\rangle_{\mathcal{H}_0^1}$$
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Wide network limit

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Theorem

For any $T > 0$,

$$\sup_{0\leq t\leq T}\mathbb{E}\left[\left\|V_t^N - V_t\right\|_{\mathcal{H}_0^1}\right] \overset{N\to\infty}{\longrightarrow} 0.$$

Convergence in time

Theorem

$$\lim_{t\to\infty} \|V_t - w_*\|_{\mathcal{H}_0^1} = 0.$$

$$\frac{\mathrm{d}V_t(x)}{\mathrm{d}t} = -\left\langle \mathcal{D}I^k(V_t), Z(x,.)\right\rangle_{\mathcal{H}_0^1}$$

Convergence in time

Theorem

$$\lim_{t\to\infty} \|V_t - w_*\|_{\mathcal{H}_0^1} = 0.$$

$$\begin{aligned}
\frac{\mathrm{d}V_t(x)}{\mathrm{d}t} &= -\left\langle \mathcal{D}I^k(V_t), Z(x,.)\right\rangle_{\mathcal{H}_0^1} \\
\frac{\mathrm{d}(V_t - w_*)(x)}{\mathrm{d}t} &= -\left\langle \mathcal{D}I^k(V_t - w_* + w_*), Z(x,.)\right\rangle_{\mathcal{H}_0^1} \\
&= -\tilde{\mathcal{T}}(V_t - w_*)(x)
\end{aligned}$$

Convergence in time

Proof: $\lim_{t\to\infty} \|V_t - w_*\|_{\mathcal{H}_0^1} = 0$.

$\tilde{\mathcal{T}}$ is a self-adjoint, positive definite trace class operator. Spectral decomposition:

$$\tilde{\mathcal{T}}\left(\tilde{e}_i\right) = \lambda_i \tilde{e}_i,$$

$\lambda_1 \geq \lambda_2 \geq ... > 0$, orthogonal basis $\{\tilde{e}_i\}_{i=1}^{\infty}$.

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$h_t^i = e^{-\lambda_i t} h_0^i$.

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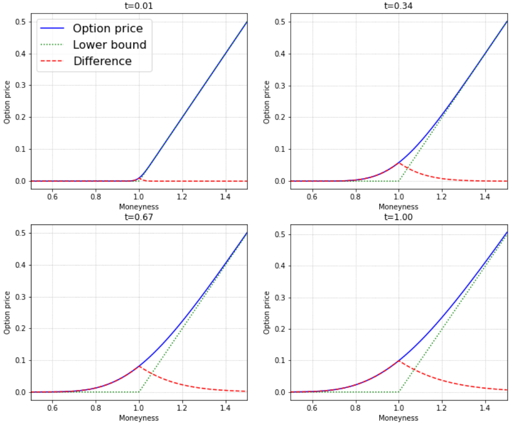
$h_t^i = e^{-\lambda_i t} h_0^i$. Parseval's identity:

$$\| V_t - w_* \|^2 = \sum_{i=1}^{\infty} \left(h_t^i\right)^2 = \sum_{i=1}^{\infty} e^{-2\lambda_i t} \left(h_0^i\right)^2 \xrightarrow{t\to\infty} 0. \qquad \square$$

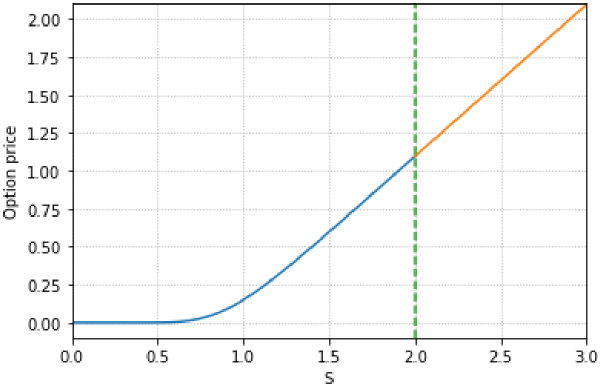
Architecture: base

No-arbitrage bound: $u(t, S) \geq S - Ke^{-rt}$



Architecture: linearization

$$u(x_p + y; \theta) = u(x_p; \theta) + y, \quad y > 0.$$



Architecture

$$
\begin{aligned}
S^1 &= \sigma_1\left(W^1\mathbf{x} + b^1\right), && \\
Z^l &= \sigma_1\left(U^{z,l}\mathbf{x} + W^{z,l}S^l + b^{z,l}\right), && l = 1, \ldots, L, \\
G^l &= \sigma_1\left(U^{g,l}\mathbf{x} + W^{g,l}S^1 + b^{g,l}\right), && l = 1, \ldots, L, \\
R^l &= \sigma_1\left(U^{r,l}\mathbf{x} + W^{r,l}S^l + b^{r,l}\right), && l = 1, \ldots, L, \\
H^l &= \sigma_1\left(U^{h,l}\mathbf{x} + W^{h,l}\left(S^l \odot R^l\right) + b^{h,l}\right), && l = 1, \ldots, L, \\
S^{l+1} &= \left(1 - G^l\right) \odot H^l + Z^l \odot S^l, && l = 1, \ldots, L, \\
f(\theta) &= \text{base} + \sigma_2\left(WS^{L+1} + b\right), && \sigma_2 > 0.
\end{aligned}
$$

Algorithm

1: Initialize $f(\theta^0; \mathbf{x}) = \Phi(S)$.

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for each time step $k = 1, ..., N_t$ **do**
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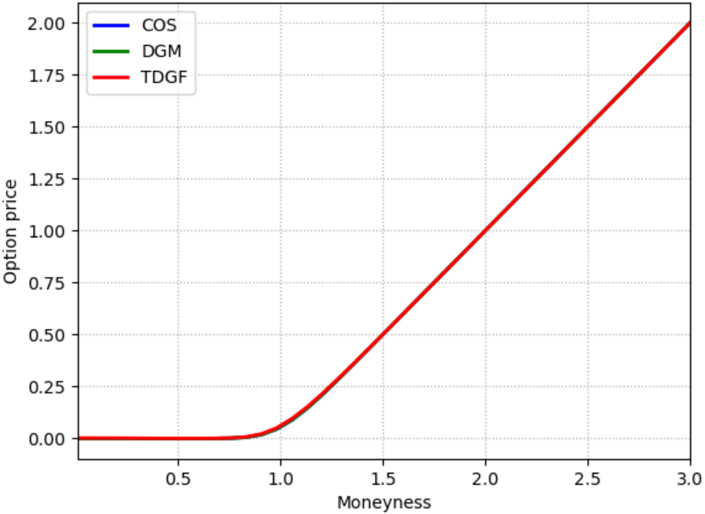
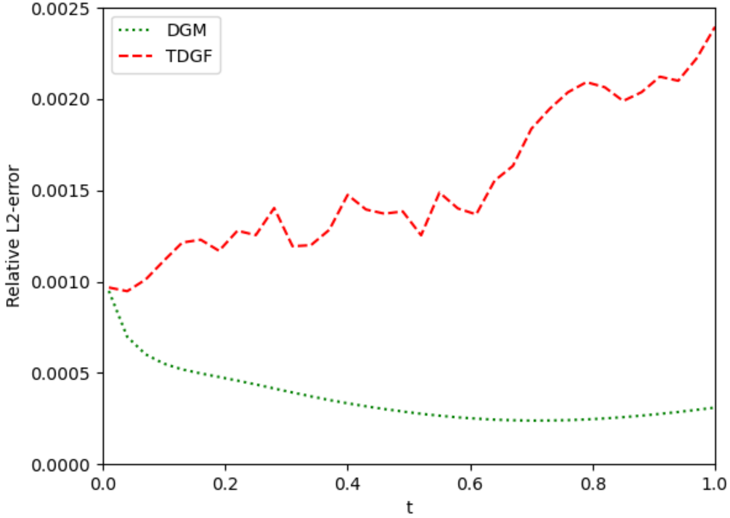
        Generate random points $\mathbf{x}^i$ for training.
        Calculate the cost functional $I^k(f(\theta_n^k; \mathbf{x}^i))$.
        Take a descent step $\theta_{n+1}^k = \theta_n^k - \alpha_n \nabla_\theta I^k(f(\theta_n^k; \mathbf{x}^i))$.
    end for
end for

Heston



COS
DGM
TDGF
Option price
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
0.5
1.0
1.5
2.0
2.5
3.0
Moneyness

Heston

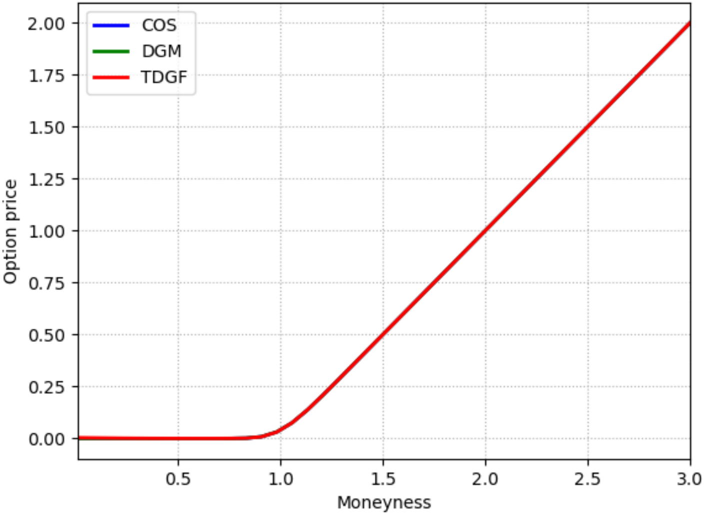


DGM
TDGF
Relative L2-error
0.0025
0.0020
0.0015
0.0010
0.0005
0.0000
0.0
0.2
0.4
0.6
0.8
1.0
t

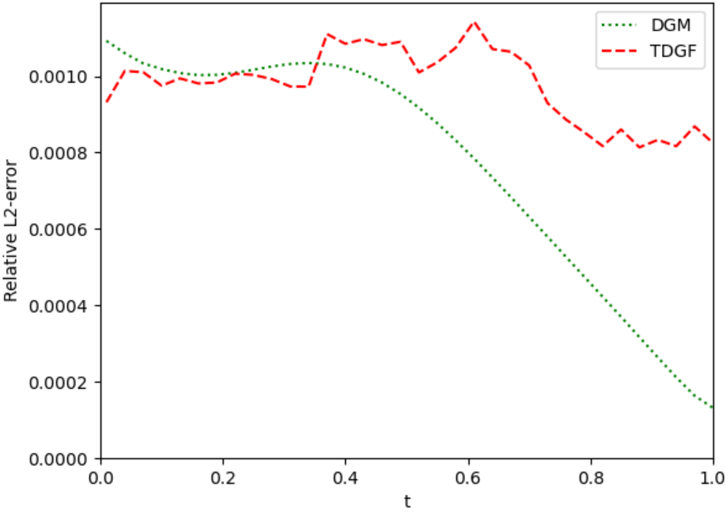
Lifted Heston

$$
\begin{aligned}
dS_t &= rS_t dt + \sqrt{V_t^n} S_t dW_t, & S_0 > 0, \\
V_t^n &= g^n(t) + \sum_{i=1}^{n} c_i^n V_t^{n,i}, & \\
dV_t^{n,i} &= -\left(\gamma_i^n V_t^{n,i} + \lambda V_t^n\right) dt + \eta \sqrt{V_t^n} dB_t, & V_0^{n,i} = 0, \\
g^n(t) &= V_0 + \lambda\theta \sum_{i=1}^{n} c_i^n \int_0^t e^{-\gamma_i^n (t-s)} ds. &
\end{aligned}
$$

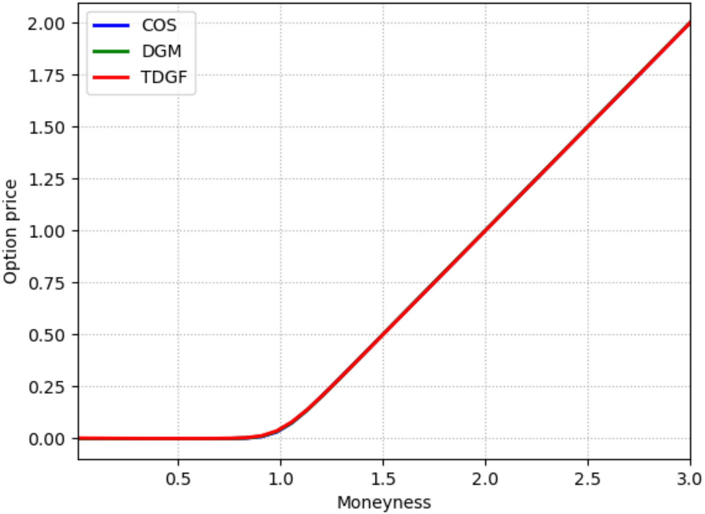
Lifted Heston, $n = 1$



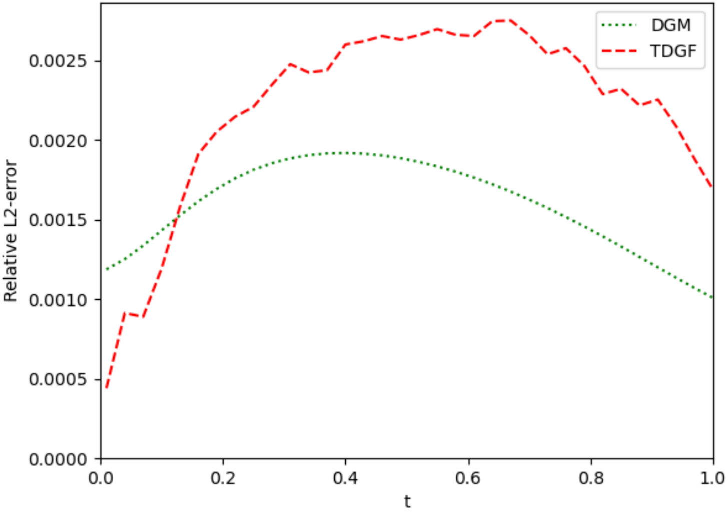
Lifted Heston, $n = 1$



Lifted Heston, $n = 20$



Lifted Heston, $n = 20$



Running times

Model	Heston	LH, n=1	LH, n=20
DGM	12.5×10^3	13.3×10^3	56.1×10^3
TDGF	6.0×10^3	6.4×10^3	7.6×10^3

Table: Training time

Running times

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DGM	12.5×10^3	13.3×10^3	56.1×10^3
TDGF	6.0×10^3	6.4×10^3	7.6×10^3

Table: Training time

Model	Heston	LH, n=1	LH, n=20
COS	0.018	8.9	10.4
DGM	0.0016	0.0034	0.0053
TDGF	0.0060	0.020	0.025

Table: Computing time

Time-Stepping Deep Gradient Flow Methods

Seminar National Technical University of Athens

Jasper Rou

June 14, 2024

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