Time-Stepping Deep Gradient Flow Methods

Seminar National Technical University of Athens

Jasper Rou
joint work with Chenguang Liu & Antonis Papapantoleon

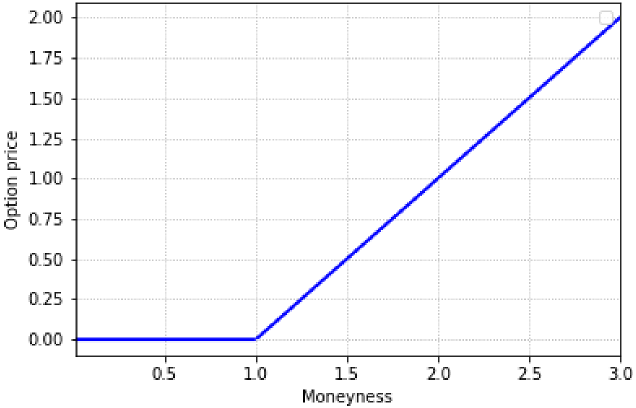
June 14, 2024

Options

A contract which gives the owner the right, but not the obligation, to buy a stock at a price K at a future time T

Pay-off

$$\Phi(S_T) = (S_T \quad K)^+$$



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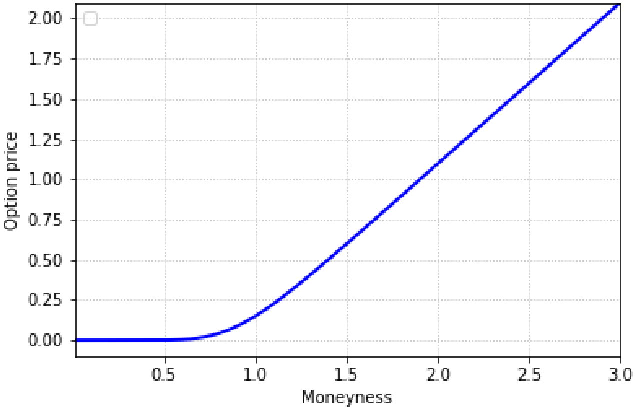

Stock price

$$dS_t = rS_t dt + \sqrt{V_t} S_t dW_t \qquad S_0 > 0$$
$$dV_t = \ (\quad V_t) dt + \ \sqrt{V_t} dB_t \qquad V_0 > 0$$

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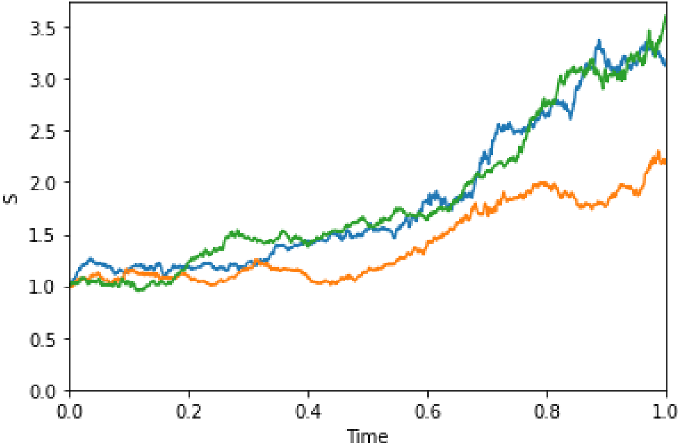

Pricing

Price of a derivative with pay-off $\Phi(S_T)$

$$u(t; S) = \mathrm{E}\left[\Phi(S_T) | S_t\right]$$

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$$\frac{\partial u}{\partial t} + \sum_{i,j=0}^{n} a^{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=0}^{n} b^{i}\frac{\partial u}{\partial x_i} - ru = 0,$$

$$u(T,x) = \Phi(x)$$

Pricing

Price of a derivative with pay-off $\Phi(S_T)$

$$u(t) = \mathbb{E}\left[e^{-r(T-t)}\Phi(S_T) | S_t \right]$$

$$\frac{\partial u}{\partial t} - \sum_{i,j=0}^{n} a^{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{i=0}^{n} b^i \frac{\partial u}{\partial x_i} + ru = 0,$$

$$u(0,x) = \Phi(x)$$

Deep Galerkin Method

$$\frac{\partial u}{\partial t} \quad \sum_{i,j=0}^{n} a^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} \quad \sum_{i=0}^{n} b^{i} \frac{\partial u}{\partial x_i} + ru = 0$$

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$$u(0,x) = \Phi(x)$$

Minimize

$$\left\| \frac{\partial u}{\partial t} - \sum_{i,j=0}^{n} a^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{i=0}^{n} b^i \frac{\partial u}{\partial x_i} + ru \right\|^2_{[0,T]\times\Omega} + \| u(0,x) - \Phi(x) \|^2_{\Omega}$$

Deep Galerkin Method

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Issue: Taking second derivative makes training in high dimensions slow

Idea

Rewrite PDE as energy minimization problem

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- Only first order derivative
- No norm

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Split in symmetric and non-symmetric part

Splitting method

$$\frac{\partial u}{\partial t} = \sum_{i,j=0}^{n} a^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=0}^{n} b^i \frac{\partial u}{\partial x_i} - ru$$

Splitting method

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \sum_{i,j=0}^{n} a^{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=0}^{n} b^i \frac{\partial u}{\partial x_i} - ru \\
&= \sum_{i,j=0}^{n} \frac{\partial}{\partial x_j}\left(a^{ij}\frac{\partial u}{\partial x_i}\right) - \sum_{i,j=0}^{n} \frac{\partial a^{ij}}{\partial x_j}\frac{\partial u}{\partial x_i} + \sum_{i=0}^{n} b^i \frac{\partial u}{\partial x_i} - ru
\end{aligned}
$$

Splitting method

$$\frac{\partial u}{\partial t} = \sum_{i,j=0}^{n} a^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=0}^{n} b^i \frac{\partial u}{\partial x_i} - ru$$

$$= \sum_{i,j=0}^{n} \frac{\partial}{\partial x_j}\left(a^{ij} \frac{\partial u}{\partial x_i}\right) - \sum_{i,j=0}^{n} \frac{\partial a^{ij}}{\partial x_j} \frac{\partial u}{\partial x_i} + \sum_{i=0}^{n} b^i \frac{\partial u}{\partial x_i} - ru$$

$$= \sum_{i,j=0}^{n} \frac{\partial}{\partial x_j}\left(a^{ij} \frac{\partial u}{\partial x_i}\right) + \sum_{i=0}^{n} \left(b^i - \sum_{j=0}^{n} \frac{\partial a^{ij}}{\partial x_j} \right) \frac{\partial u}{\partial x_i} - ru$$

Splitting method

$$\frac{\partial u}{\partial t} = \sum_{i,j=0}^{n} a^{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=0}^{n} b^i \frac{\partial u}{\partial x_i} - ru$$

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$$= \sum_{i,j=0}^{n} \frac{\partial}{\partial x_j}\left(a^{ij}\frac{\partial u}{\partial x_i}\right) - \sum_{i=0}^{n}\left(\sum_{j=0}^{n}\frac{\partial a^{ij}}{\partial x_j} - b^i\right)\frac{\partial u}{\partial x_i} - ru$$

$$= \nabla\cdot(A\nabla u) - ru - F(u)$$

$$F(u) = \mathbf{b}\cdot\nabla u$$

Example: Heston model

$$
\begin{aligned}
dS_t &= rS_t dt + \sqrt{V_t} S_t dW_t \qquad && S_0 > 0 \\
dV_t &= (\quad V_t) dt + \sqrt{V_t} dB_t \qquad && V_0 > 0
\end{aligned}
$$

Example: Heston model

$$dS_t = rS_t dt + \sqrt{V_t} S_t dW_t \qquad S_0 > 0$$

$$dV_t = \ (\quad V_t) dt + \sqrt{V_t} dB_t \qquad V_0 > 0$$

$$\frac{\partial u}{\partial t} = rS \frac{\partial u}{\partial S} + \ (\quad V) \frac{\partial u}{\partial V} + \frac{1}{2} S^2 V \frac{\partial^2 u}{\partial S^2} + \frac{1}{2} \ {}^2 V \frac{\partial^2 u}{\partial V^2} + \quad SV \frac{\partial^2 u}{\partial S \partial V} \quad ru$$

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$$\frac{@u}{@t} = rS\frac{@u}{@S} + (\quad V)\frac{@u}{@V} + \frac{1}{2}S^2V\frac{@^2u}{@S^2} + \frac{1}{2}\ {}^2V\frac{@^2u}{@V^2} + \quad SV\frac{@^2u}{@S@V} \quad ru$$

$$= rS\frac{@u}{@S} + (\quad V)\frac{@u}{@V} + \frac{@}{@S}\quad \frac{1}{2}S^2V\frac{@u}{@S} \quad SV\frac{@u}{@S}$$

$$+ \frac{@}{@V}\quad \frac{1}{2}\ {}^2V\frac{@u}{@V} \quad \frac{1}{2}\ {}^2\frac{@u}{@V} + \frac{@}{@S}\quad \frac{1}{2}\quad SV\frac{@u}{@V} \quad \frac{1}{2}\quad V\frac{@u}{@V}$$

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$$+ \frac{\partial}{\partial V}\quad \frac{1}{2}\ ^2V\frac{\partial u}{\partial V} \quad \frac{1}{2}\ ^2\frac{\partial u}{\partial V} + \frac{\partial}{\partial S}\quad \frac{1}{2}\quad SV\frac{\partial u}{\partial V} \quad \frac{1}{2}\quad V\frac{\partial u}{\partial V}$$

$$+ \frac{\partial}{\partial V}\quad \frac{1}{2}\quad SV\frac{\partial u}{\partial S} \quad \frac{1}{2}\quad S\frac{\partial u}{\partial S} \quad ru$$

$$= r \quad \frac{1}{2}\quad \begin{matrix} S^2V & SV \\ SV & ^2V \end{matrix} \quad r\,u \quad ru$$

$$\begin{matrix} SV & rS + \frac{1}{2}\quad S \\ (V \quad) + \frac{1}{2}\quad V + \frac{1}{2}\ ^2 \end{matrix} \quad r\,u$$

Time Deep Gradient Flow

$$\begin{cases} u_t - \nabla \cdot (A \nabla u) + ru + F(u) = 0 & (t, \mathbf{x}) \in [0, T] \times \Omega \\ u(0, \mathbf{x}) = \Phi(\mathbf{x}) & \mathbf{x} \in \Omega \end{cases}$$

Time Deep Gradient Flow

$$
\begin{cases}
u_t \quad r \quad (A\, r\, u) + ru + F(u) = 0 & (t;\mathbf{x})\ 2\ [0;T] \quad \Omega \\
u(0;\mathbf{x}) = \Phi(\mathbf{x}) & \mathbf{x}\ 2\ \Omega
\end{cases}
$$

Divide $[0;T]$ in intervals $(t_{k \quad 1}; t_k]$ with $h = t_k \quad t_{k \quad 1}$

$$
\frac{U^k \quad U^{k \quad 1}}{h} \quad r \quad A\, r\, U^k \quad + rU^k + F \quad U^{k \quad 1} \quad = 0
$$

$$
U^0 = \Phi
$$

Time Deep Gradient Flow

$$
\begin{cases}
u_t - \nabla \cdot (A \nabla u) + ru + F(u) = 0 & (t, \mathbf{x}) \in [0, T] \times \Omega \\
u(0, \mathbf{x}) = \Phi(\mathbf{x}) & \mathbf{x} \in \Omega
\end{cases}
$$

Divide $[0, T]$ in intervals $(t_{k-1}, t_k]$ with $h = t_k - t_{k-1}$

$$
\frac{U^k - U^{k-1}}{h} - \nabla \cdot \left(A \nabla U^k \right) + rU^k + F\left(U^{k-1} \right) = 0
$$

$$
U^0 = \Phi
$$

Theorem (Akrivis and Crouzeix 2004)

There exists a constant C independent of h and k such that

$$
\max_{0 \le k \le N} \left\| u(t_k) - U^k \right\| \le Ch
$$

Time Deep **Gradient Flow**

$$\frac{U^k \quad U^{k\quad 1}}{h} \quad r \quad A\, r\, U^k \quad + rU^k + F(U^{k\quad 1}) = 0$$

Time Deep **Gradient Flow**

$$\frac{U^k \quad U^{k \;\; 1}}{h} \quad r \quad A\, r\, U^k \quad + rU^k + F(U^{k \;\; 1}) = 0$$

$$I^k(u) = \frac{1}{2} \;\; u \quad U^{k \;\; 1} \;\; {}^2 + h \int_\Omega \frac{1}{2} \;\; (\, r\, u)^T A\, r\, u + ru^2 \quad + F \quad U^{k \;\; 1} \quad udx$$

Time Deep **Gradient Flow**

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Theorem

The minimizer $w \in H_0^1 \; \mathbb{R}^d$ of I^k is the unique solution U^k.

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$$i^k(\quad) = I^k(w \;\; + \;\; v)$$

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Since w minimizes I^k, $\quad = 0$ minimizes i^k.

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Proof.

$$i^k(\;) = I^k(w \; + \; v)$$

Since w minimizes I^k, $\; = 0$ minimizes i^k.

$$0 = \; i^k \;{}^{0}(0)$$

$$= \int_{\mathrm{R}^d} w \quad U^{k \;\; 1} + h \quad r \quad (A\, r\, w\;) + rw \; + F \quad U^{k \;\; 1} \qquad vdx:$$

□

Time **Deep** Gradient Flow

Definition (Activation function)

An activation function is a function $\sigma : \mathbb{R}^d \to \mathbb{R}$ such that $\sigma \in C_c^1(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d} \sigma(x)\,\mathrm{d}x \neq 0$.

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Definition (Neural network)

$$C^n(\ ) = \left\{ (x) : \mathbb{R}^d \to \mathbb{R} : (x) = \sum_{i=1}^{n} {}_i \, ({}_i x + c_i) \right\};$$

$$C(\ ) = \bigcup_{n \geq 1} C^n(\ )$$

Time **Deep** Gradient Flow

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Definition (Neural network)

$$C^n(\sigma) = \left\{ \psi(x) : \mathbb{R}^d \to \mathbb{R} : \psi(x) = \sum_{i=1}^{n} \eta_i \, \sigma(\omega_i x + c_i) \right\},$$

$$C(\sigma) = \cup_{n \geq 1} C^n(\sigma)$$

Theorem

$C(\sigma)$ is dense in $H_0^1(\mathbb{R}^d)$.

Convergence of the minimizer

Theorem

Let w_m be a sequence in $H_0^1(\mathbb{R}^d)$ and w^* the minimizer of I^k.

$$\lim_{m\to\infty} \|w_m - w^*\|_{H_0^1} = 0 \iff \lim_{m\to\infty} I^k(w_m) = I^k(w^*)$$

$$I^k(u) = \frac{1}{2}\left\|u - U^{k-1}\right\|^2 + h\int_{\mathbb{R}^d} \frac{1}{2}\left((\nabla u)^T A \nabla u + ru^2\right) + F\left(U^{k-1}\right) u \,\mathrm{d}x$$

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Let w_m be a sequence in $H_0^1 \ \mathrm{R}^d$ and w the minimizer of I^k.

$$\lim_{m\ !\ 1} kw_m \quad w\ k_{H_0^1} = 0 \qquad (\ ) \qquad \lim_{m\ !\ 1} I^k(w_m) = I^k(w\ )$$

$$I^k(u) = \frac{1}{2}\ u \quad U^{k\ \ 1}\ ^2 + h \int_{\mathrm{R}^d} \frac{1}{2}\ (\ r\,u)^T A\, r\,u + ru^2 \ + F\ \ U^{k\ \ 1}\ \ u\mathrm{d}x$$

$$=: L^k(u) + G^k(u);$$

$$L^k(u) = \frac{1}{2} kuk^2 + \frac{h}{2} \int_{\mathrm{R}^d} (\ r\,u)^T A\, r\,u + ru^2 \mathrm{d}x;$$

$$G^k(u) = \quad \mathrm{D}\ u; U^{k\ \ 1}\ \mathrm{E} + \frac{1}{2}\ U^{k\ \ 1}\ ^2 + h \int_{\mathrm{R}^d} F\ \ U^{k\ \ 1}\ \ u\mathrm{d}x$$

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Let w_m be a sequence in $H_0^1 \;\; \mathrm{R}^d$ and w the minimizer of I^k.

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Proof.

$=)$ $\quad I^k$ is continuous.

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Let w_m be a sequence in $H_0^1\ \mathrm{R}^d$ and w the minimizer of I^k.

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Convergence of the minimizer

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Let w_m be a sequence in $H_0^1(\mathbb{R}^d)$ and w_* the minimizer of I^k.

$$\lim_{m\to\infty} \|w_m - w_*\|_{H_0^1} = 0 \iff \lim_{m\to\infty} I^k(w_m) = I^k(w_*)$$

Proof.

$\Longrightarrow$ I^k is continuous.

$\Longleftarrow$ w_m * w_*. So $G^n[w_m] \to G^n[w_*]$.

Since $I^k(w_m) \to I^k(w_*)$, $L^k(w_m) \to L^k(w_*)$.

Convergence of the minimizer

Theorem

Let w_m be a sequence in $H_0^1(\mathbb{R}^d)$ and w^* the minimizer of I^k.

$$\lim_{m \to \infty} \|w_m - w^*\|_{H_0^1} = 0 \iff \lim_{m \to \infty} I^k(w_m) = I^k(w^*)$$

Proof.

$\Longrightarrow$ I^k is continuous.

$\Longleftarrow$ $w_m \rightharpoonup w^*$. So $G^n[w_m] \to G^n[w^*]$.

Since $I^k(w_m) \to I^k(w^*)$, $L^k(w_m) \to L^k(w^*)$.

$$\frac{1+hr}{2}\|w_m - w^*\|^2 + \frac{h}{2}\left\|\sqrt{A}\,\nabla(w_m - w^*)\right\|^2 \to 0.$$

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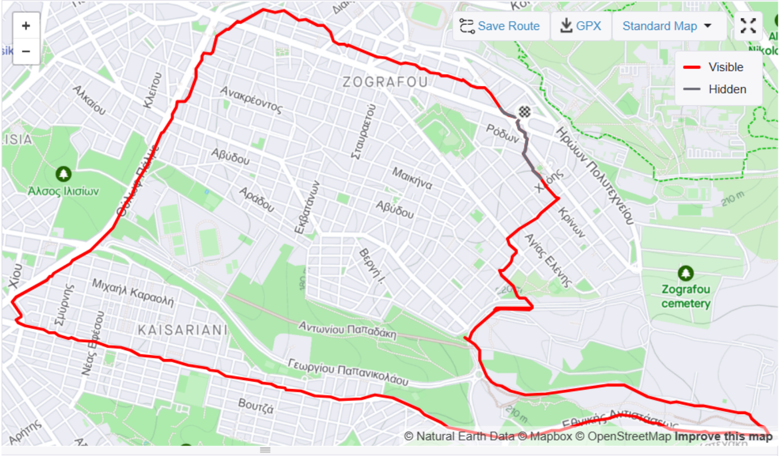
$\|w_m - w^*\|_{H_0^1} \to 0.$ □

Intermezzo: Gradient Descent



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Convergence when training

Neural network:

$$V_t^N \; {}^N; x \;=\; V^N \; {}_t^N; x \;=\; N \; \sum_{i=1}^{N} \; {}^i \; {}^i x + c^i \; ;$$

$${}^N = \; {}^i; \; {}^i; c^i \; {}_{i=1}^{N}, \; \tfrac{1}{2} < \; < \; 1.$$

Convergence when training

Neural network:

$$V_t^N \quad {}^N; x \quad = \; V^N \quad {}_t^N; x \quad = \; N \quad \sum_{i=1}^{N} \quad {}^i \qquad {}^i x + c^i \quad ;$$

$${}^N = \quad {}^i; \quad {}^i; c^i \quad {}_{i=1}^{N}, \; \tfrac{1}{2} < \quad < \quad 1.$$

$$V_t^N \xrightarrow{N \to 1} \quad V_t \xrightarrow{t \to 1} \quad w$$

Gradient Descent

$$V^{N} \quad {}^{N}; x \quad = N \sum_{i=1}^{N} \quad {}^{i} \quad {}^{i} x + c^{i} \quad ;$$

${}^{N} = \quad {}^{i}; \quad {}^{i}; c^{i} \quad {}_{i=1}^{N}, \ \frac{1}{2} < \quad < \quad 1.$

Gradient Descent

$$V^N \quad {}^N; x \quad = N \quad \sum_{i=1}^{N} \quad {}^i \quad {}^i x + c^i \quad ;$$

${}^N = \quad {}^i; \quad {}^i; c^i \;{}_{i=1}^{N},\ \frac{1}{2} < \quad < \quad 1. \quad {}_N = N^2 \quad {}^1$

$$\frac{d \;{}_t^N}{dt} = \quad {}_N r \quad I^k \quad V^N \quad {}_t^N; x$$

Gradient Descent

$$V^N \quad {}^N; x \quad = N \sum_{i=1}^{N} \quad {}^i \quad {}^i x + c^i \quad ;$$

$${}^N = \quad {}^i; \quad {}^i; c^i \quad {}_{i=1}^{N}, \ \tfrac{1}{2} < \quad < \quad 1. \quad {}_N = N^2 \quad {}^1$$

$$\frac{d \ {}^N_t}{dt} = \quad {}_N r \quad I^k \quad V^N \quad {}^N_t; x$$

$$\frac{dV^N_t(x)}{dt} = r \quad V^N \quad {}^N_t; x \quad \frac{d \ {}^N_t}{dt}$$

$$= \quad {}_N r \quad V^N \quad {}^N_t; x \quad r \quad I^k \quad V^N \quad {}^N_t; x$$

Wide network limit

$$\frac{dV_t^N(x)}{dt} = \quad {}_N r \quad V^N \quad {}^N_t; x \quad r \quad l^k \quad V^N \quad {}^N_t; x$$

Wide network limit

$$\frac{dV_t^N(x)}{dt} = \overset{N}{D} r \quad V^N \quad {}^N_t ; x \quad r \quad I^k \quad V^N \quad {}^N_t ; x \; E$$

$$= \; D I^k \quad V_t^N \quad ; Z_t^N(x; :) \quad {}_{H_0^1}$$

$$Z_t^N(x; y) = N^{\;1} \sum_{i=1}^{N} r \;_{; ; \; c} \; {}^i_t \quad {}^i_t x + c^i_t \quad r \;_{; ; \; c} \; {}^i_t \quad {}^i_t y + c^i_t$$

Wide network limit

$$\frac{dV_t^N(x)}{dt} = \left\langle \nabla_N V^N \; {}_t^N ; x \quad \nabla \; I^k \; V^N \; {}_t^N ; x \right\rangle$$

$$= \left\langle DI^k \; V_t^N \; ; Z_t^N(x; :) \right\rangle_{H_0^1}$$

$$\frac{dV_t(x)}{dt} = \left\langle DI^k(V_t) ; Z(x; :) \right\rangle_{H_0^1}$$

$$Z_t^N(x;y) = N^{1} \sum_{i=1}^{N} \nabla_{;;c} \; {}_t^i \quad {}_t^i x + c_t^i \quad \nabla_{;;c} \; {}_t^i \quad {}_t^i y + c_t^i$$

$$Z(x;y) = E \; \nabla_{;;c} \; {}_0^1 \quad {}_0^1 x + c_0^1 \quad \nabla_{;;c} \; {}_0^1 \quad {}_0^1 y + c_0^1$$

Wide network limit

$$\frac{dV_t^N(x)}{dt} = \left\langle DI^k\left(V_t^N\right); Z_t^N(x;:)\right\rangle_{H_0^1}$$

$$\frac{dV_t(x)}{dt} = \left\langle DI^k(V_t); Z(x;:)\right\rangle_{H_0^1}$$

$$Z_t^N(x;y) = N^{-1}\sum_{i=1}^{N} \nabla_{;;c}\left(_t^i \quad _t^{i;N}x + c_t^{i;N}\right)\nabla_{;;c}\left(_t^i \quad _t^{i;N}y + c_t^{i;N}\right)$$

$$Z(x;y) = E\left[\nabla_{;;c}\left(_0^1 \quad _0^1x + c_0^1\right)\nabla_{;;c}\left(_0^1 \quad _0^1y + c_0^1\right)\right]$$

Wide network limit

$$\frac{dV_t^N(x)}{dt} = -\left\langle DI^k\left(V_t^N\right), Z_t^N(x,\cdot)\right\rangle_{H_0^1}$$

$$\frac{dV_t(x)}{dt} = -\left\langle DI^k(V_t), Z(x,\cdot)\right\rangle_{H_0^1}$$

$$Z_t^N(x,y) = N^{-1}\sum_{i=1}^{N} \nabla_{\theta,c}\,\sigma\left(\theta_t^{i,N}x + c_t^{i,N}\right)\cdot\nabla_{\theta,c}\,\sigma\left(\theta_t^{i,N}y + c_t^{i,N}\right)$$

$$Z(x,y) = \mathbb{E}\left[\nabla_{\theta,c}\,\sigma\left(\theta_0^1 x + c_0^1\right)\cdot\nabla_{\theta,c}\,\sigma\left(\theta_0^1 y + c_0^1\right)\right]$$

Theorem

For any $T > 0$,

$$\sup_{0\le t\le T}\mathbb{E}\left\|V_t^N - V_t\right\|_{H_0^1} \xrightarrow{N\to\infty} 0.$$

Convergence in time

Theorem

$$\lim_{t \to \infty} \| V_t - w \|_{H_0^1} = 0 .$$

$$\frac{dV_t(x)}{dt} = - \left\langle DI^k(V_t), Z(x; \cdot) \right\rangle_{H_0^1}$$

Convergence in time

Theorem

$$\lim_{t \to \infty} \| V_t \quad w \|_{H^1_0} = 0 .$$

$$\frac{dV_t(x)}{dt} = \left\langle DI^k(V_t) ; Z(x; \cdot) \right\rangle_{H^1_0}$$

$$\frac{d(V_t \quad w)(x)}{dt} = \left\langle DI^k(V_t \quad w + w) ; Z(x; \cdot) \right\rangle_{H^1_0}$$

$$= \tilde{T}(V_t \quad w)(x)$$

Convergence in time

Proof: $\lim_{t\to\infty} \|V_t - w\|_{H^1_0} = 0$.

$\tilde{T}$ is a self-adjoint, positive definite trace class operator. Spectral decomposition:

$$\tilde{T}(e_i) = \lambda_i e_i;$$

$\lambda_1 \geq \lambda_2 \geq \dots > 0$; orthogonal basis $\{e_i\}_{i=1}^{\infty}$.

Convergence in time

Proof: $\lim_{t \to \infty} \quad \| V_t \quad w \|_{H^1_0} = 0.$

$\tilde{T}$ is a self-adjoint, positive definite trace class operator. Spectral decomposition:

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$$_1 \qquad _2 \qquad ::: > 0; \text{ orthogonal basis } \{e_i\}_{i=1}^{\infty}.$$

$$\frac{dh^i_t}{dt} := \frac{\langle d(V_t \quad w); e_i \rangle}{dt} = \qquad \tilde{T}(V_t \quad w); e_i \quad = \qquad V_t \quad w; \tilde{T}(e_i)$$
$$= \qquad {}_i h^i_t :$$

Convergence in time

Proof: $\lim_{t\to\infty} \|V_t - w\|_{H^1_0} = 0$.

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$$\tilde{T}(e_i) = \lambda_i e_i;$$

$\lambda_1 \geq \lambda_2 \geq \dots > 0$; orthogonal basis $\{e_i\}_{i=1}^{\infty}$.

$$\frac{dh_t^i}{dt} := \frac{\langle d(V_t - w), e_i\rangle}{dt} = -\langle \tilde{T}(V_t - w), e_i\rangle = -\langle V_t - w, \tilde{T}(e_i)\rangle$$
$$= -\lambda_i h_t^i.$$

$h_t^i = e^{-\lambda_i t} h_0^i$.

Convergence in time

Proof: $\lim_{t\to\infty} \|V_t - w^*\|_{H_0^1} = 0$.

$\tilde{T}$ is a self-adjoint, positive definite trace class operator. Spectral decomposition:

$$\tilde{T}(\tilde{e}_i) = \lambda_i \tilde{e}_i,$$

$\lambda_1 \geq \lambda_2 \geq \dots > 0$, orthogonal basis $\{\tilde{e}_i\}_{i=1}^{\infty}$.

$$\frac{\mathrm{d}h_t^i}{\mathrm{d}t} := \frac{\mathrm{d}\langle (V_t - w^*), \tilde{e}_i\rangle}{\mathrm{d}t} = \langle -\tilde{T}(V_t - w^*), \tilde{e}_i\rangle = -\langle V_t - w^*, \tilde{T}(\tilde{e}_i)\rangle = -\lambda_i h_t^i.$$

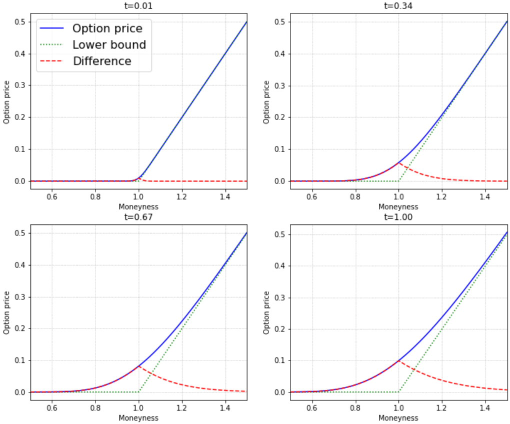
$h_t^i = e^{-\lambda_i t} h_0^i$. Parseval's identity:

$$\|V_t - w^*\|^2 = \sum_{i=1}^{\infty} \left(h_t^i\right)^2 = \sum_{i=1}^{\infty} e^{-2\lambda_i t} \left(h_0^i\right)^2 \xrightarrow{t\to\infty} 0.$$

□

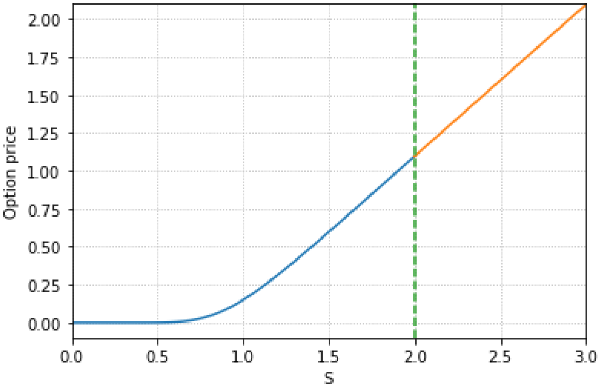
Architecture: base

No-arbitrage bound: $u(t;S) \quad S \quad Ke^{\quad rt}$



Architecture: linearization

$$u(x_p + y; \quad) = u(x_p; \quad) + y; \quad y > 0:$$



Architecture

$$
\begin{aligned}
S^1 &= {}_1 \;\; W^1\mathbf{x} + b^1 \;\; ; \\
Z^l &= {}_1 \;\; U^{z;l}\mathbf{x} + W^{z;l}S^l + b^{z;l} \;\; ; && l = 1; :::; L; \\
G^l &= {}_1 \;\; U^{g;l}\mathbf{x} + W^{g;l}S^1 + b^{g;l} \;\; ; && l = 1; :::; L; \\
R^l &= {}_1 \;\; U^{r;l}\mathbf{x} + W^{r;l}S^l + b^{r;l} \;\; ; && l = 1; :::; L; \\
H^l &= {}_1 \;\; U^{h;l}\mathbf{x} + W^{h;l} \;\; S^l \;\;\; R^l \;\; + b^{h;l} \;\; ; && l = 1; :::; L; \\
S^{l+1} &= 1 \;\;\; G^l \;\;\;\;\; H^l + Z^l \;\;\; S^l; && l = 1; :::; L; \\
f(\;\;) &= \text{base} + {}_2 \;\; WS^{L+1} + b \;\; ; && {}_2 > 0:
\end{aligned}
$$

Algorithm

1: Initialize $f(\ ^{0}; \mathbf{x}) = \Phi(S)$.

Algorithm

Initialize $f(\ ^{0}; \mathbf{x}) = \Phi(S)$.
for each time step $k = 1; :::; N_t$ **do**
    Initialize $\ _{0}^{k} = \ ^{k\ \ 1}$.

Algorithm

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    Generate random points $\mathbf{x}^i$ for training.

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        Calculate the cost functional $I^k(f(\ ^{k}_{n};\mathbf{x}^i))$.

Algorithm

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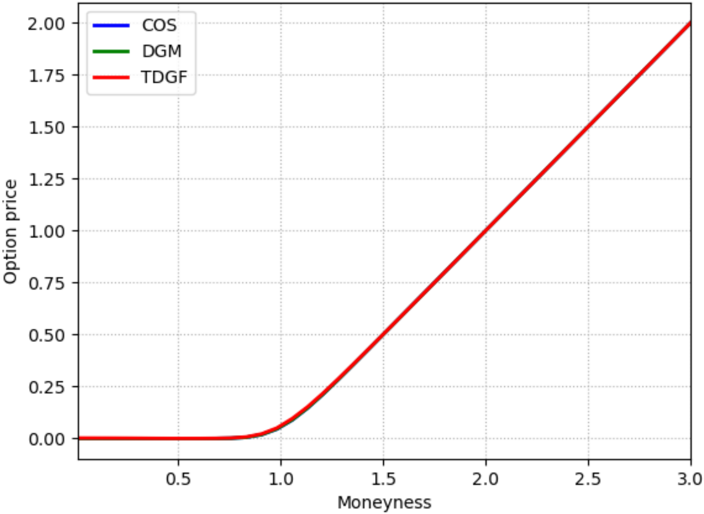
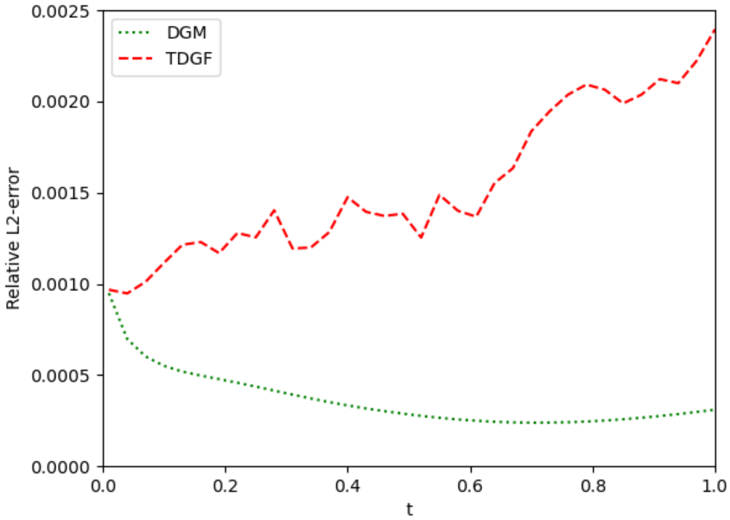
for each time step $k = 1; :::; N_t$ **do**
    Initialize $^{k}_{0} = \ ^{k\ \ 1}$.
    for each sampling stage n **do**
        Generate random points $\mathbf{x}^i$ for training.
        Calculate the cost functional $I^k(f(\ ^{k}_{n};\mathbf{x}^i))$.
        Take a descent step $^{k}_{n+1} = \ ^{k}_{n} \quad _{n}\, r \ \ I^k(f(\ ^{k}_{n};\mathbf{x}^i))$.
    end for
end for

Heston


COS
DGM
TDGF
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
Option price
0.5
1.0
1.5
2.0
2.5
3.0
Moneyness

Heston

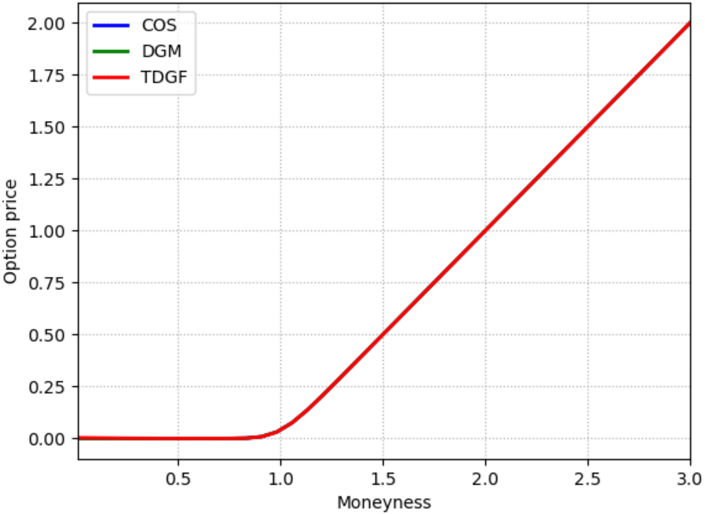


DGM
TDGF
Relative L2-error
0.0025
0.0020
0.0015
0.0010
0.0005
0.0000
0.0
0.2
0.4
0.6
0.8
1.0
t

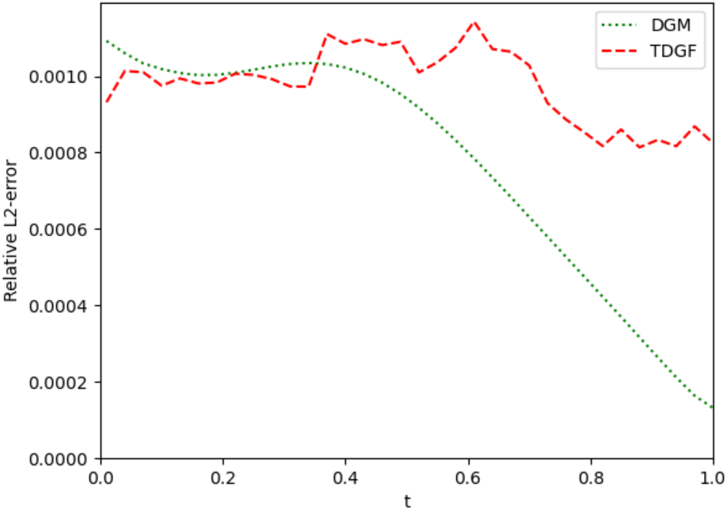
Lifted Heston

$$
\begin{aligned}
dS_t &= rS_t dt + \sqrt{V_t^n} S_t dW_t, & S_0 > 0, \\
V_t^n &= g^n(t) + \sum_{i=1}^{n} c_i^n V_t^{n,i}, \\
dV_t^{n,i} &= \left(-\gamma_i^n V_t^{n,i} - \lambda V_t^n\right) dt + \nu \sqrt{V_t^n} dB_t, & V_0^{n,i} = 0, \\
g^n(t) &= V_0 + \lambda\theta \sum_{i=1}^{n} c_i^n \int_0^t e^{-\gamma_i^n (t-s)} ds.
\end{aligned}
$$

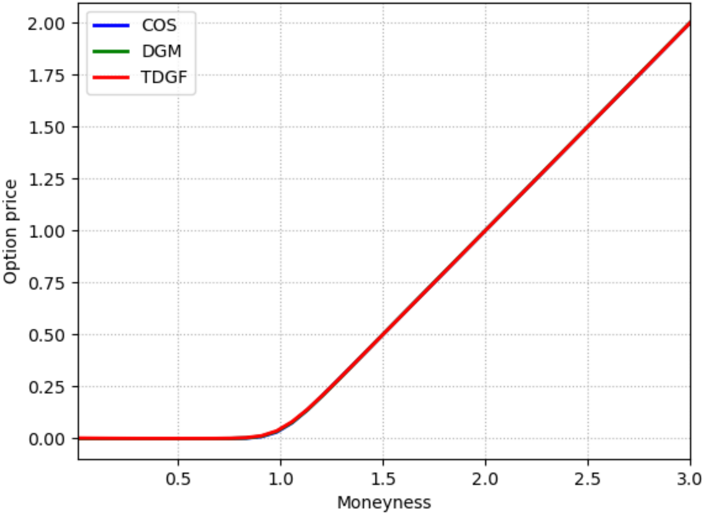
Lifted Heston, $n = 1$



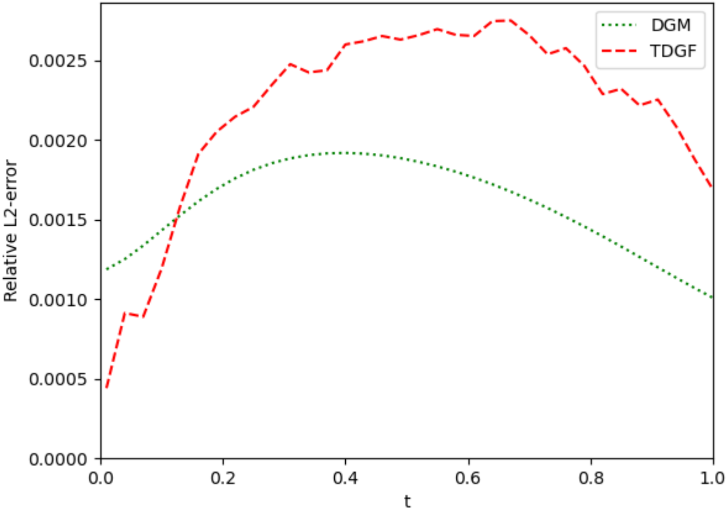
Lifted Heston, $n = 1$



Lifted Heston, $n = 20$



Lifted Heston, $n = 20$



Running times

Model	Heston	LH, n=1	LH, n=20
DGM	12:5 10^3	13:3 10^3	56:1 10^3
TDGF	6:0 10^3	6:4 10^3	7:6 10^3

Table: Training time

Running times

Model	Heston	LH, n=1	LH, n=20
DGM	12:5 10^3	13:3 10^3	56:1 10^3
TDGF	6:0 10^3	6:4 10^3	7:6 10^3

Table: Training time

Model	Heston	LH, n=1	LH, n=20
COS	0.018	8.9	10.4
DGM	0.0016	0.0034	0.0053
TDGF	0.0060	0.020	0.025

Table: Computing time

Time-Stepping Deep Gradient Flow Methods

Seminar National Technical University of Athens

Jasper Rou

June 14, 2024

j.g.rou@tudelft.nl www.jasperrou.nl