Deep Gradient Flow Methods for Option Pricing in Diffusion Models

SIAM Conference on Financial Mathematics and Engineering

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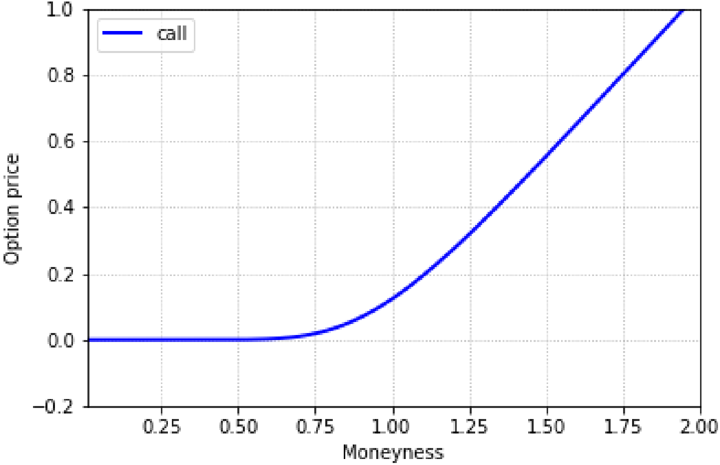
Option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Option price
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2.00

Pricing

Feynman-Kac formula:

$$\frac{\partial u}{\partial t} + \sum_{i,j=0}^{n} a^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{i=0}^{n} b^i \frac{\partial u}{\partial x_i} - ru = 0,$$

$$u(T) = \Phi(S_T)$$

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Can we solve this PDE using a neural network?

Deep Galerkin Method [1]

$$\frac{\partial u}{\partial t} + \sum_{i,j=0}^{n} a^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{i=0}^{n} b^i \frac{\partial u}{\partial x_i} - ru = 0,$$

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[1] Justin Sirignano and Konstantinos Spiliopoulos (2018). "DGM: A deep learning algorithm for solving partial differential equations". In: *Journal of computational physics* 375, pp. 1339–1364

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Minimize

$$\left\| \frac{\partial u}{\partial t} + \sum_{i,j=0}^{n} a^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{i=0}^{n} b^i \frac{\partial u}{\partial x_i} - ru \right\|^2_{[0,T]\times\Omega} + \| u(T) - \Phi(S_T) \|^2_{\Omega}.$$

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Splitting method

$$\frac{\partial u}{\partial t} = -\sum_{i,j=0}^{n} a^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=0}^{n} b^i \frac{\partial u}{\partial x_i} + ru$$

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$$= -\sum_{i,j=0}^{n} \frac{\partial}{\partial x_j}\left(a^{ij}\frac{\partial u}{\partial x_i}\right) + \sum_{i,j=0}^{n} \frac{\partial a^{ij}}{\partial x_j}\frac{\partial u}{\partial x_i} + \sum_{i=0}^{n} b^i \frac{\partial u}{\partial x_i} + ru$$

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\begin{aligned}
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&= -\sum_{i,j=0}^{n} \frac{\partial}{\partial x_j}\left(a^{ij}\frac{\partial u}{\partial x_i}\right) + \sum_{i=0}^{n}\left(b^i + \sum_{j=0}^{n}\frac{\partial a^{ij}}{\partial x_j}\right)\frac{\partial u}{\partial x_i} + ru.
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&= -\sum_{i,j=0}^{n} \frac{\partial}{\partial x_j}\left(a^{ij}\frac{\partial u}{\partial x_i}\right) + \sum_{i=0}^{n}\left(b^i + \sum_{j=0}^{n}\frac{\partial a^{ij}}{\partial x_j}\right)\frac{\partial u}{\partial x_i} + ru. \\
&= -\nabla \cdot (A\nabla u) + ru + F(u), \\
F(u) &= \mathbf{b} \cdot \nabla u.
\end{aligned}
$$

Time Deep Nitsche Method [2]

$$\begin{cases} u_\tau - \nabla \cdot (A \nabla u) + ru + F(u) = 0, & (\tau, \mathbf{x}) \in [0, T] \times \Omega, \\ u(0, \mathbf{x}) = \Phi(\mathbf{x}), & \mathbf{x} \in \Omega, \\ u(\tau, \mathbf{x}) = g_D(t, \mathbf{x}), & (\tau, \mathbf{x}) \in [0, T] \times \Gamma_D, \\ \mathbf{n} \cdot A \nabla u(\tau, \mathbf{x}) = g_N(\tau, \mathbf{x}), & (\tau, \mathbf{x}) \in [0, T] \times \Gamma_N. \end{cases}$$

[2] Emmanuil H Georgoulis, Michail Loulakis, and Asterios Tsiourvas (2023). "Discrete gradient flow approximations of high dimensional evolution partial differential equations via deep neural networks". In: *Communications in Nonlinear Science and Numerical Simulation* 117, p. 106893

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- Divide $[0, T]$ in intervals $(\tau_{k-1}, \tau_k]$ with $h = \tau_k - \tau_{k-1}$
- Seek approximations $f^k(\mathbf{x}; \theta)$ such that

$$\frac{f^k - f^{k-1}}{h} - \nabla \cdot \left(A\nabla f^k\right) + rf^k + F\left(f^{k-1}\right) = 0.$$

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$$\frac{f^k - f^{k-1}}{h} - \nabla \cdot \left(A \nabla f^k \right) + r f^k + F = 0.$$

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$$0 = \int_\Omega \left(-\nabla \cdot (A \nabla u) + r u + F \right) v d\mathbf{x} = i'(0),$$

$$i(\tau) = I(u + \tau v)$$

$$I(u) = \int_\Omega \frac{1}{2} \left((\nabla u)^T A \nabla u + r u^2 \right) + F u dx - \int_{\Gamma_N} g_N u ds$$

$$- \int_{\Gamma_D} \mathbf{n} \cdot A \nabla u (u - g_D) ds.$$

Algorithm

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4: Initialize $\theta_0^k = \theta^{k-1}$.

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    for each sampling stage **do**
        Generate random points $\mathbf{x}^i$ for training.

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    for each sampling stage **do**
        Generate random points $\mathbf{x}^i$ for training.
        Calculate the cost functional $L(f(\theta_n^k; \mathbf{x}^i))$.
        Take a descent step $\theta_{n+1}^k = \theta_n^k - \alpha_n \nabla_\theta L(f(\theta_n^k; \mathbf{x}^i))$.
    end for
end for

Black-Scholes

$$dS_t = rS_t dt + \sigma S_t dW_t, \quad S_0 > 0,$$

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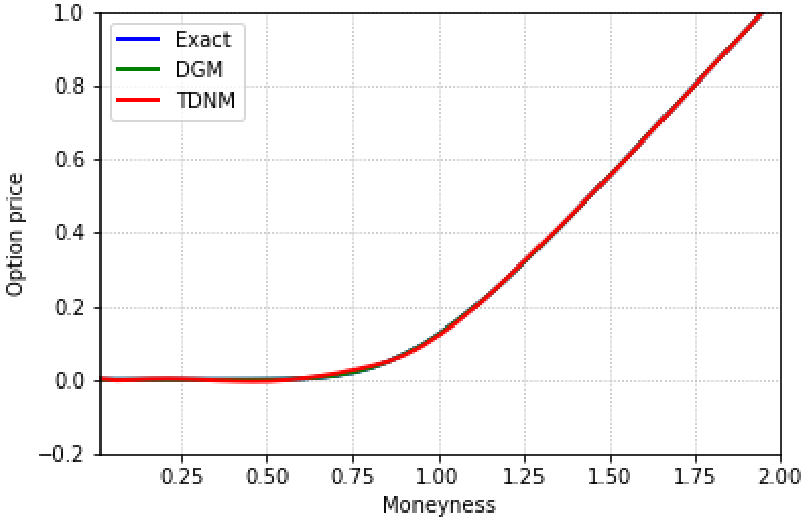
Black-Scholes



Heston

$$
\begin{aligned}
dS_t &= rS_t dt + \sqrt{V_t} S_t dW_t, \quad S_0 > 0, \\
dV_t &= \kappa(\theta - V_t) dt + \eta \sqrt{V_t} dB_t, \quad V_0 > 0.
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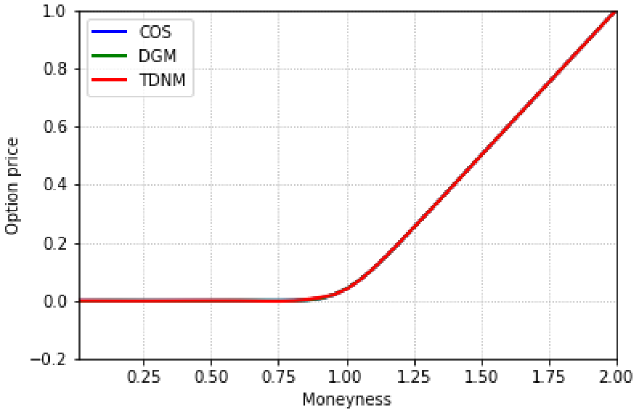


Lifted Heston [3]

$$dS_t = rS_t dt + \sqrt{V_t^n} S_t dW_t, \qquad S_0 > 0,$$

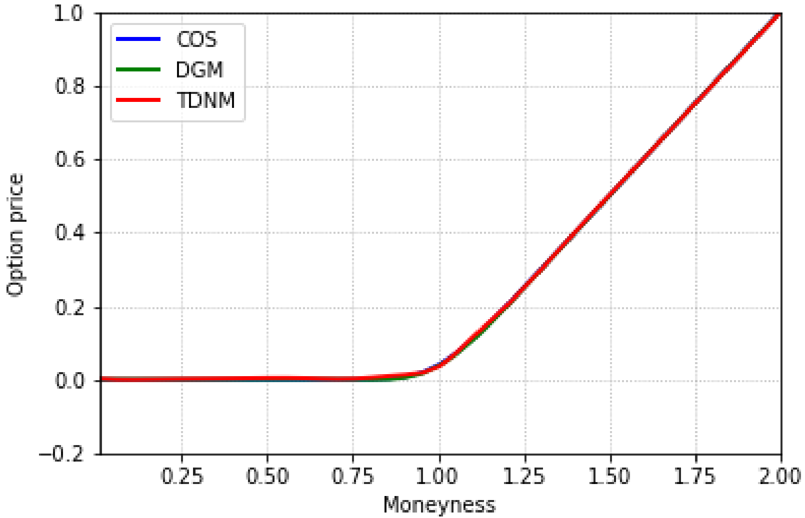
$$V_t^n = g^n(t) + \sum_{i=1}^{n} c_i^n V_t^{n,i},$$

$$dV_t^{n,i} = -\left(\gamma_i^n V_t^{n,i} + \lambda V_t^n\right) dt + \eta \sqrt{V_t^n} dB_t, \quad V_0^{n,i} = 0,$$

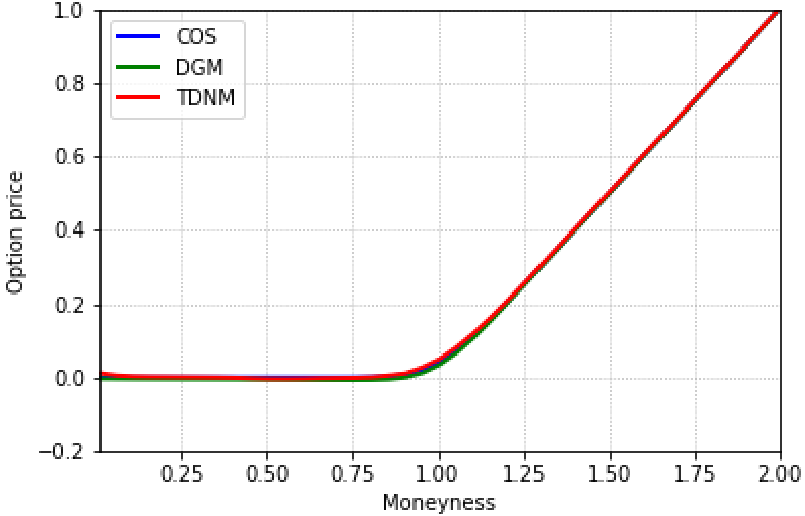
$$g^n(t) = V_0 + \lambda\theta \sum_{i=1}^{n} c_i^n \int_0^t e^{-\gamma_i^n (t-s)} ds,$$

[3] Eduardo Abi Jaber (2019). "Lifting the Heston model". In: *Quantitative Finance* 19.12, pp. 1995–2013

Lifted Heston, $n = 1$



Lifted Heston, $n = 5$



Running times

Method	Black-Scholes	Heston	LH, n=1	LH, n=5
DGM	0.34	0.65	2.0	4.4
TDNM	0.55	0.71	0.76	1.2

Table: Training time (10^4 seconds)

Running times

Method	Black-Scholes	Heston	LH, n=1	LH, n=5
DGM	0.34	0.65	2.0	4.4
TDNM	0.55	0.71	0.76	1.2

Table: Training time (10^4 seconds)

Method	Black-Scholes	Heston	LH, n=1	LH, n=5
Exact/COS	0.00051	0.013	6.9	6.9
DGM	0.011	0.011	0.011	0.011
TDNM	0.027	0.036	0.053	0.052

Table: Computing time (seconds)

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