Error Analysis of Deep PDE Solvers for Option Pricing

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Option Pricing

$$\frac{\partial u}{\partial t} - \nabla \cdot (A \nabla u) + \mathbf{b} \cdot \nabla u + ru = 0$$
$$u(0, \mathbf{x}) = \Psi(\mathbf{x})$$

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$$u(0, \mathbf{x}) = \Psi(\mathbf{x})$$

$$f(t, \mathbf{x}; \theta) \approx u(t, \mathbf{x})$$

Black–Scholes

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t$$

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Black–Scholes

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t$$

$$\begin{aligned} 0 &= \frac{\partial u}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 u}{\partial S^2} - rS\frac{\partial u}{\partial S} + ru \\ &= \frac{\partial u}{\partial t} - \frac{\partial}{\partial S}\left(\frac{1}{2}\sigma^2 S^2 \frac{\partial u}{\partial S}\right) + \sigma^2 S \frac{\partial u}{\partial S} - rS\frac{\partial u}{\partial S} + ru \end{aligned}$$

Black–Scholes

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t \mathrm{d}W_t$$

$$\begin{aligned}
0 &= \frac{\partial u}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 u}{\partial S^2} - rS\frac{\partial u}{\partial S} + ru \\
&= \frac{\partial u}{\partial t} - \frac{\partial}{\partial S}\left(\frac{1}{2}\sigma^2 S^2 \frac{\partial u}{\partial S}\right) + \sigma^2 S \frac{\partial u}{\partial S} - rS\frac{\partial u}{\partial S} + ru \\
&= \frac{\partial u}{\partial t} - \frac{\partial u}{\partial S}\left(\frac{1}{2}\sigma^2 S^2 \frac{\partial u}{\partial S}\right) + (\sigma^2 - r)S\frac{\partial u}{\partial S} + ru
\end{aligned}$$

Heston model

$$dS_t = rS_t dt + \sqrt{V_t} S_t dW_t \qquad S_0 > 0$$
$$dV_t = \kappa(\theta - V_t)dt + \eta\sqrt{V_t} dB_t \qquad V_0 > 0$$

Heston model

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$$dV_t = \kappa(\theta - V_t)dt + \eta\sqrt{V_t} dB_t \qquad V_0 > 0$$

$$0 = \frac{\partial u}{\partial t} - rS\frac{\partial u}{\partial S} - \kappa(\theta - V)\frac{\partial u}{\partial V} - \frac{1}{2}S^2 V\frac{\partial^2 u}{\partial S^2} - \frac{1}{2}\eta^2 V\frac{\partial^2 u}{\partial V^2} - \rho\eta SV\frac{\partial^2 u}{\partial S \partial V} + ru$$

Heston model

$$dS_t = rS_t dt + \sqrt{V_t} S_t dW_t \qquad S_0 > 0$$
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$$\begin{aligned}
0 =& \frac{\partial u}{\partial t} - rS\frac{\partial u}{\partial S} - \kappa(\theta - V)\frac{\partial u}{\partial V} - \frac{1}{2}S^2 V\frac{\partial^2 u}{\partial S^2} - \frac{1}{2}\eta^2 V\frac{\partial^2 u}{\partial V^2} - \rho\eta SV\frac{\partial^2 u}{\partial S\partial V} + ru \\
=& \frac{\partial u}{\partial t} - rS\frac{\partial u}{\partial S} - \kappa(\theta - V)\frac{\partial u}{\partial V} - \frac{\partial}{\partial S}\left(\frac{1}{2}S^2 V\frac{\partial u}{\partial S}\right) + SV\frac{\partial u}{\partial S} \\
& - \frac{\partial}{\partial V}\left(\frac{1}{2}\eta^2 V\frac{\partial u}{\partial V}\right) + \frac{1}{2}\eta^2\frac{\partial u}{\partial V} - \frac{\partial}{\partial S}\left(\frac{1}{2}\rho\eta SV\frac{\partial u}{\partial V}\right) + \frac{1}{2}\rho\eta V\frac{\partial u}{\partial V} \\
& - \frac{\partial}{\partial V}\left(\frac{1}{2}\rho\eta SV\frac{\partial u}{\partial S}\right) + \frac{1}{2}\rho\eta S\frac{\partial u}{\partial S} + ru
\end{aligned}$$

Heston model

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$$\begin{aligned}
0 =& \frac{\partial u}{\partial t} - rS\frac{\partial u}{\partial S} - \kappa(\theta - V)\frac{\partial u}{\partial V} - \frac{1}{2}S^2 V\frac{\partial^2 u}{\partial S^2} - \frac{1}{2}\eta^2 V\frac{\partial^2 u}{\partial V^2} - \rho\eta SV\frac{\partial^2 u}{\partial S\partial V} + ru \\
=& \frac{\partial u}{\partial t} - rS\frac{\partial u}{\partial S} - \kappa(\theta - V)\frac{\partial u}{\partial V} - \frac{\partial}{\partial S}\left(\frac{1}{2}S^2 V\frac{\partial u}{\partial S}\right) + SV\frac{\partial u}{\partial S} \\
& - \frac{\partial}{\partial V}\left(\frac{1}{2}\eta^2 V\frac{\partial u}{\partial V}\right) + \frac{1}{2}\eta^2\frac{\partial u}{\partial V} - \frac{\partial}{\partial S}\left(\frac{1}{2}\rho\eta SV\frac{\partial u}{\partial V}\right) + \frac{1}{2}\rho\eta V\frac{\partial u}{\partial V} \\
& - \frac{\partial}{\partial V}\left(\frac{1}{2}\rho\eta SV\frac{\partial u}{\partial S}\right) + \frac{1}{2}\rho\eta S\frac{\partial u}{\partial S} + ru \\
=& \frac{\partial u}{\partial t} - \nabla\cdot\left(\frac{1}{2}\begin{bmatrix} S^2 V & \rho\eta SV \\ \rho\eta SV & \eta^2 V\end{bmatrix}\nabla u\right) + \begin{bmatrix} SV - rS + \frac{1}{2}\rho\eta S \\ \kappa(V-\theta) + \frac{1}{2}\rho\eta V + \frac{1}{2}\eta^2\end{bmatrix}\cdot\nabla u + ru
\end{aligned}$$

Deep **Galerkin** Method

$$\frac{\partial u}{\partial t} - \nabla \cdot (A \nabla u) + \mathbf{b} \cdot \nabla u + ru = 0$$

$$u(0, \mathbf{x}) = \Psi(\mathbf{x})$$

Deep **Galerkin** Method

$$\frac{\partial u}{\partial t} - \nabla \cdot (A \nabla u) + \mathbf{b} \cdot \nabla u + ru = 0$$
$$u(0, \mathbf{x}) = \Psi(\mathbf{x})$$

Minimize

$$\left\| \frac{\partial u}{\partial t} - \nabla \cdot (A \nabla u) + \mathbf{b} \cdot \nabla u + ru \right\|^2 + \| u(0, \mathbf{x}) - \Psi(\mathbf{x}) \|^2 .$$

Deep Galerkin Method

$$\left\| \frac{\partial u}{\partial t} - \nabla \cdot (A \nabla u) + \mathbf{b} \cdot \nabla u + ru \right\|^2 + \| u(0, \mathbf{x}) - \Psi(\mathbf{x}) \|^2 .$$

Deep Galerkin Method

$$\left\| \frac{\partial u}{\partial t} - \nabla \cdot (A \nabla u) + \mathbf{b} \cdot \nabla u + ru \right\|^2 + \| u(0, \mathbf{x}) - \Psi(\mathbf{x}) \|^2 .$$

$$\begin{aligned} L(\theta; t, \mathbf{x}) =& \frac{T}{M_1} \sum_{m=1}^{M_1} \left[\partial_t f(t, \mathbf{x}_m; \theta) - \nabla \cdot (A \nabla f(t, \mathbf{x}_m; \theta)) + \mathbf{b} \cdot \nabla f(t, \mathbf{x}_m; \theta) + r f(t, \mathbf{x}_m; \theta) \right]^2 \\ &+ \frac{1}{M_2} \sum_{m=1}^{M_2} \left[f(0, \mathbf{x}_m; \theta) - \Psi(\mathbf{x}_m) \right]^2 . \end{aligned}$$

Algorithm: DGM

Initialize θ_0.
for each sampling stage $n = 1, ..., N$ **do**
    Generate M random points $(t_m, \mathbf{x}_m)$ for training.
    Calculate the cost functional $L(\theta_n; t, \mathbf{x})$ for the selected points.
    Take a descent step $\theta_{n+1} = \theta_n - \alpha \nabla_\theta L(\theta_n; t, \mathbf{x})$.
end for

Time Deep Gradient Flow

$$\frac{\partial u}{\partial t} - \nabla \cdot (A \nabla u) + \mathbf{b} \cdot \nabla u + ru = 0$$
$$u(0, \mathbf{x}) = \Psi(\mathbf{x})$$

Time Deep Gradient Flow

$$\frac{\partial u}{\partial t} - \nabla \cdot (A \nabla u) + \mathbf{b} \cdot \nabla u + ru = 0$$
$$u(0, \mathbf{x}) = \Psi(\mathbf{x})$$

- Divide $[0, T]$ in K intervals $(t_{k-1}, t_k]$ with $h = t_k - t_{k-1}$, $U^0 = \Psi$

Time Deep Gradient Flow

$$\frac{\partial u}{\partial t} - \nabla \cdot (A \nabla u) + \mathbf{b} \cdot \nabla u + ru = 0$$
$$u(0, \mathbf{x}) = \Psi(\mathbf{x})$$

- Divide $[0, T]$ in K intervals $(t_{k-1}, t_k]$ with $h = t_k - t_{k-1}$, $U^0 = \Psi$

$$\frac{U^k - U^{k-1}}{h} \quad -\nabla \cdot \left(A \nabla U^k\right) \quad +\mathbf{b} \cdot \nabla U^{k-1} \quad +rU^k = 0$$

Time Deep Gradient Flow

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- Divide $[0, T]$ in K intervals $(t_{k-1}, t_k]$ with $h = t_k - t_{k-1}$, $U^0 = \Psi$

$$\frac{U^k - U^{k-1}}{h} - \nabla \cdot \left(A \nabla U^k\right) + \mathbf{b} \cdot \nabla U^{k-1} + rU^k = 0$$
$$\frac{\frac{3}{2} U^k - 2U^{k-2} + \frac{1}{2} U^{k-1}}{h} - \nabla \cdot \left(A \nabla U^k\right) + \mathbf{b} \cdot \left(2 \nabla U^{k-1} - \nabla U^{k-2}\right) + rU^k = 0$$

Time Deep **Gradient Flow**

$$\frac{U^k - U^{k-1}}{h} - \nabla \cdot \left(A \nabla U^k\right) + \mathbf{b} \cdot \nabla U^{k-1} + rU^k = 0$$

$$\frac{U^k - \frac{4}{3}U^{k-2} + \frac{1}{3}U^{k-1}}{\frac{2}{3}h} - \nabla \cdot \left(A \nabla U^k\right) + \mathbf{b} \cdot \left(2\nabla U^{k-1} - \nabla U^{k-2}\right) + rU^k = 0$$

Time Deep **Gradient Flow**

$$\frac{U^k - U^{k-1}}{h} - \nabla \cdot \left(A \nabla U^k\right) + \mathbf{b} \cdot \nabla U^{k-1} + rU^k = 0$$

$$\frac{U^k - \frac{4}{3}U^{k-2} + \frac{1}{3}U^{k-1}}{\frac{2}{3}h} - \nabla \cdot \left(A \nabla U^k\right) + \mathbf{b} \cdot \left(2\nabla U^{k-1} - \nabla U^{k-2}\right) + rU^k = 0$$

$$I_1^k(u) = \frac{1}{2}\left\| u - U^{k-1} \right\|^2 + h \left(\int \frac{1}{2}\left((\nabla u)^{\mathsf{T}} A \nabla u + ru^2 \right) + \left(\mathbf{b} \cdot \nabla U^{k-1}\right) u \mathrm{d}\mathbf{x} \right),$$

$$I_2^k(u) = \frac{1}{2}\left\| u - \frac{4}{3}U^{k-1} + \frac{1}{3}U^{k-2} \right\|^2 + \frac{2h}{3} \left(\int \frac{1}{2}\left((\nabla u)^{\mathsf{T}} A \nabla u + ru^2 \right) + \mathbf{b} \cdot \left(2\nabla U^{k-1} - \nabla U^{k-2}\right) u \mathrm{d}\mathbf{x} \right),$$

Time **Deep** Gradient Flow

$$I_n^k(u) = \frac{1}{2} \left\| u + \sum_{j=1}^{n} \alpha_n^j U^{k-j} \right\|^2 + \beta_n h \left(\int \frac{1}{2} \left((\nabla u)^\mathsf{T} A \nabla u + r u^2 \right) + \left(\mathbf{b} \cdot \sum_{j=1}^{n} \gamma_n^j \nabla U^{k-j} \right) u \mathrm{d}\mathbf{x} \right)$$

Time **Deep** Gradient Flow

$$I_n^k(u) = \frac{1}{2}\left\| u + \sum_{j=1}^{n} \alpha_n^j U^{k-j} \right\|^2 + \beta_n h \left(\int \frac{1}{2}\left((\nabla u)^\mathsf{T} A \nabla u + r u^2 \right) + \left(\mathbf{b} \cdot \sum_{j=1}^{n} \gamma_n^j \nabla U^{k-j} \right) u \mathrm{d}\mathbf{x} \right)$$

$$\begin{aligned} L_n^k(\theta; \mathbf{x}) = & \frac{1}{2M} \sum_{m=1}^{M} \left(f^k(\mathbf{x}_m; \theta) + \sum_{j=1}^{n} \alpha_n^j f^{k-j}(\mathbf{x}_m) \right)^2 \\ & + \frac{\beta_n h}{M} \sum_{m=1}^{M} \left[\frac{1}{2}\left(\left(\nabla f^k(\mathbf{x}_m; \theta) \right)^\mathsf{T} A \nabla f^k(\mathbf{x}_m; \theta) + r \left(f^k(\mathbf{x}_m; \theta) \right)^2 \right) \right. \\ & \left. + \left(\mathbf{b} \cdot \sum_{j=1}^{n} \gamma_n^j \nabla f^{k-j}(\mathbf{x}_m) \right) f^k(\mathbf{x}_m; \theta) \right]. \end{aligned}$$

Algorithm: TDGF

Initialize θ_0^0.
Set $f^0(\mathbf{x};\theta) = \Psi(\mathbf{x})$.
for each time step $k = 1, \ldots, K$ **do**
    Initialize $\theta_0^k = \theta^{k-1}$.
    for each sampling stage $n = 1, ..., N$ **do**
        Generate M random points $\mathbf{x}_m$ for training.
        Calculate the cost functional $L^k(\theta_n^k;\mathbf{x})$ for the selected points.
        Take a descent step $\theta_{n+1}^k = \theta_n^k - \alpha \nabla_\theta L^k(\theta_n^k;\mathbf{x})$.
    end for
end for

Architecture

$$X^0 = \sigma_1\left(W^0\mathbf{x} + b^0\right),$$

$$X^l = \sigma_1\left(W^l X^{l-1} + b^1\right), \qquad l = 1, \dots, L,$$

$$f(\mathbf{x};\theta) = \left(S - K\mathrm{e}^{-rt}\right)^+ + \sigma_2\left(WX^L + b\right),$$

$$\left(W^1, b^1\right) \in \left(\mathbb{R}^{D\times d}, \mathbb{R}^D\right)$$

$$\left(W^l, b^l\right) \in \left(\mathbb{R}^{D\times D}, \mathbb{R}^D\right)$$

$$\left(W, b\right) \in \left(\mathbb{R}^{1\times D}, \mathbb{R}\right)$$

Parameters

$$\left(W^1, b^1\right) \in \left(\mathbb{R}^{D\times d}, \mathbb{R}^{D}\right)$$

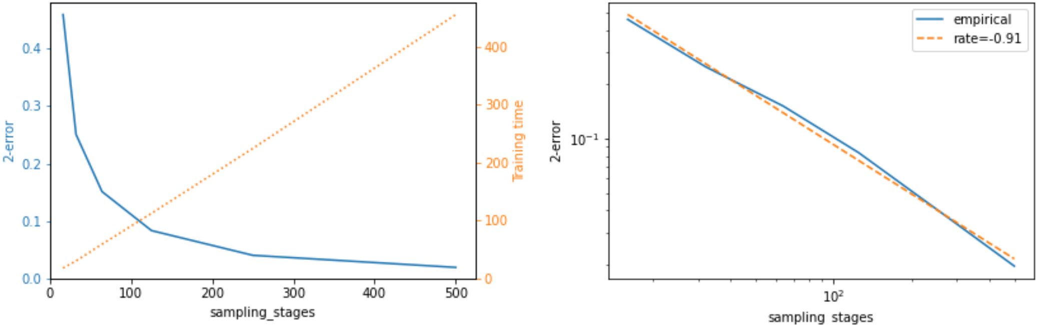
$$\left(W^l, b^l\right) \in \left(\mathbb{R}^{D\times D}, \mathbb{R}^{D}\right)$$

$$\left(W, b\right) \in \left(\mathbb{R}^{1\times D}, \mathbb{R}\right)$$

$$\theta = \left(W^1, b^1, W^l, b^l, W, b\right), \quad l = 1, \ldots, L$$

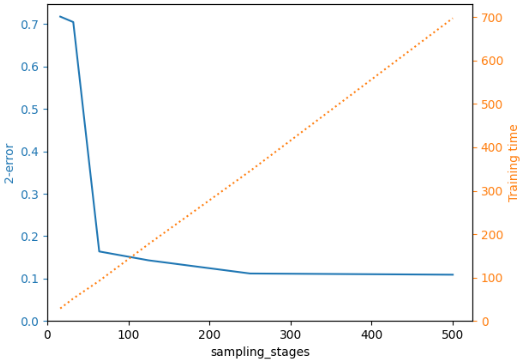
- L: layers
- D: nodes per layer

Sampling stages: TDGF, BS



(a) Linear scale (b) Logarithmic scale

Sampling stages: TDGF, Heston



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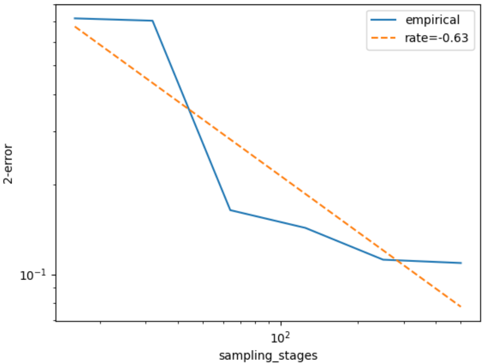
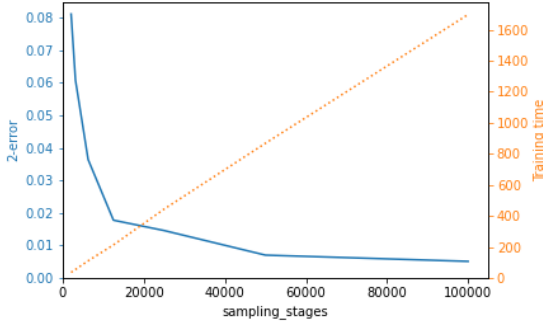


(b) Logarithmic scale

Sampling stages: DGM, BS



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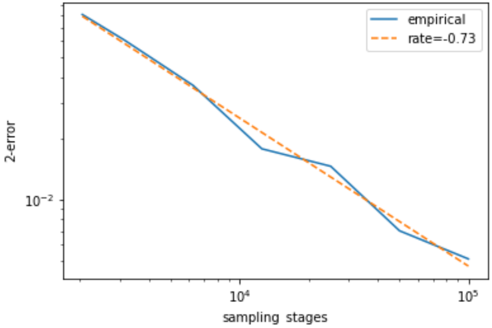
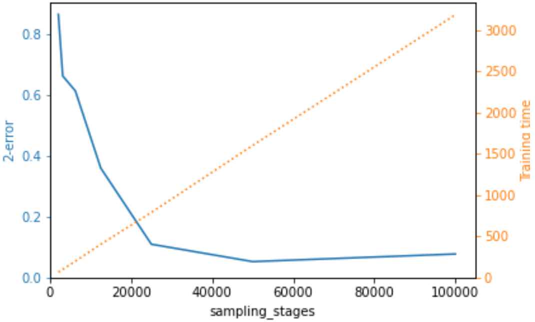


(b) Logarithmic scale

Sampling stages: DGM, Heston



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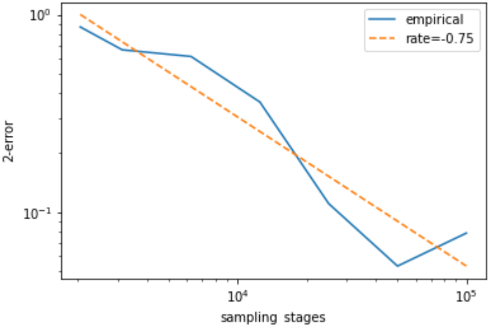
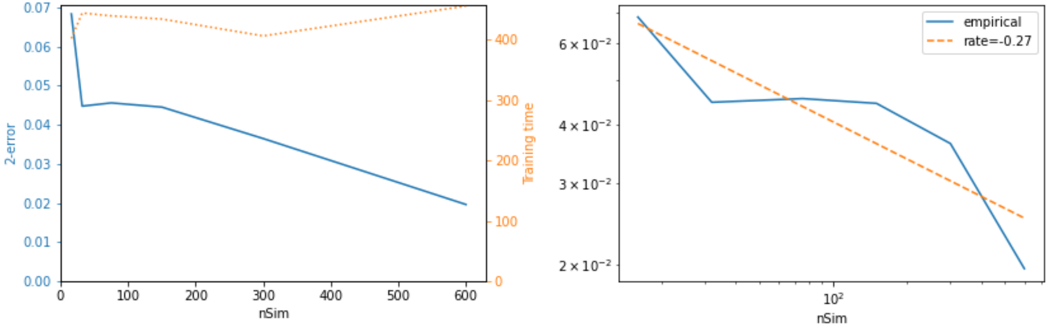


(b) Logarithmic scale

Samples: TDGF, BS



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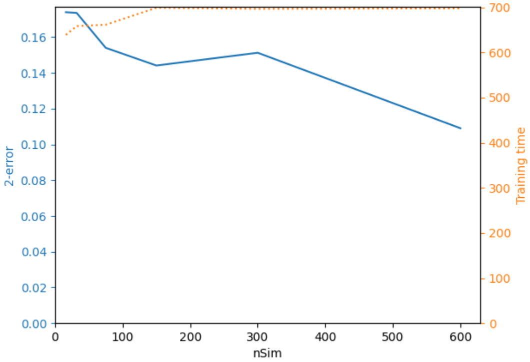
Samples: TDGF, Heston



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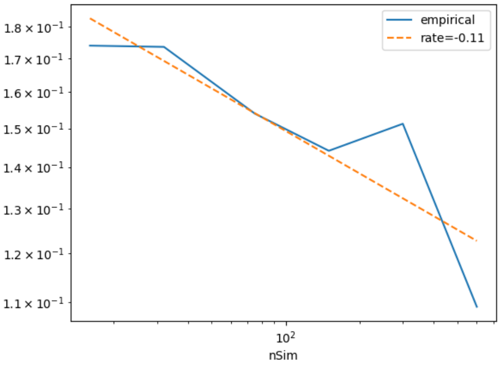

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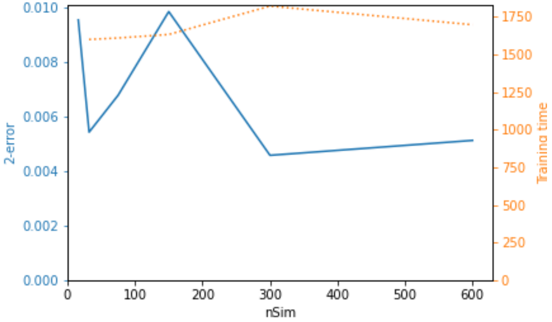

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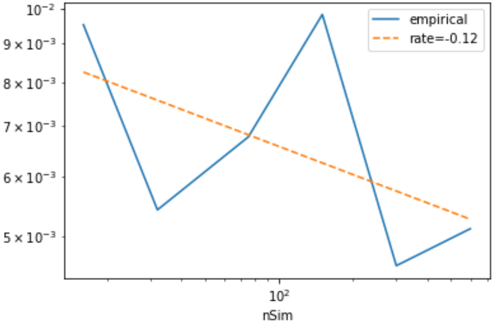

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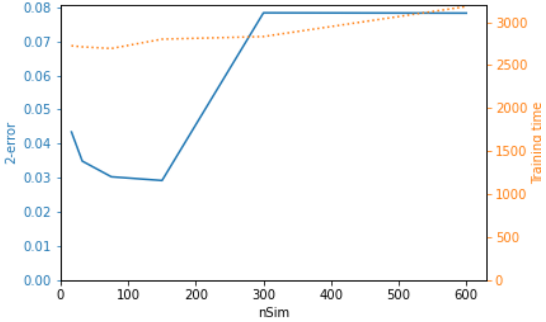

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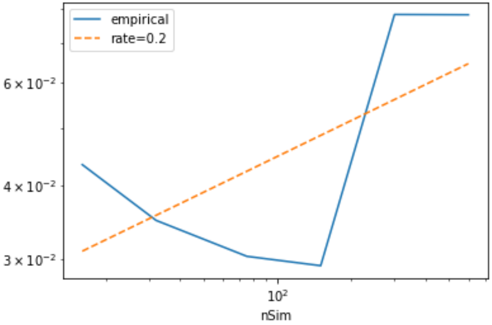
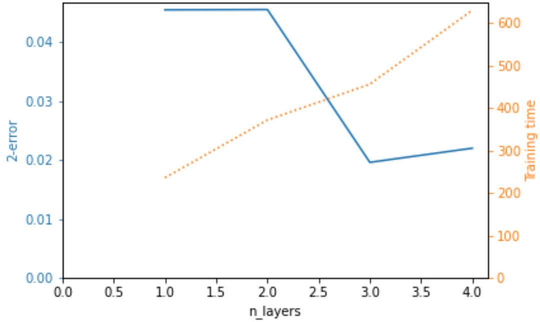


(b) Logarithmic scale

Layers: TDGF, BS



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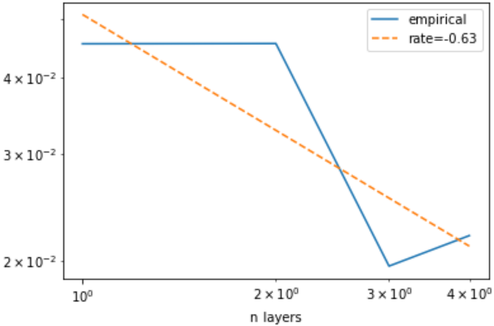

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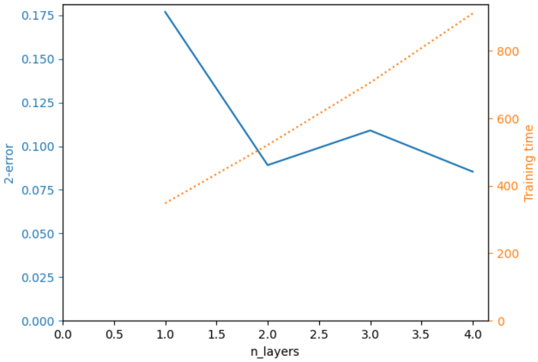
Layers: TDGF, Heston



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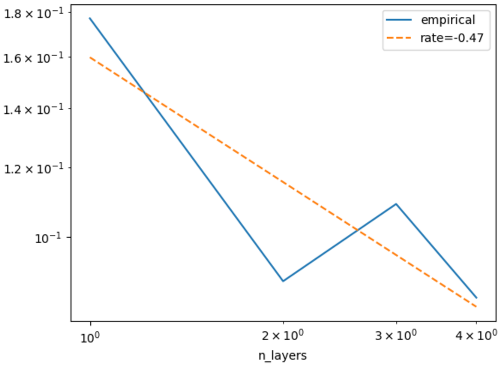

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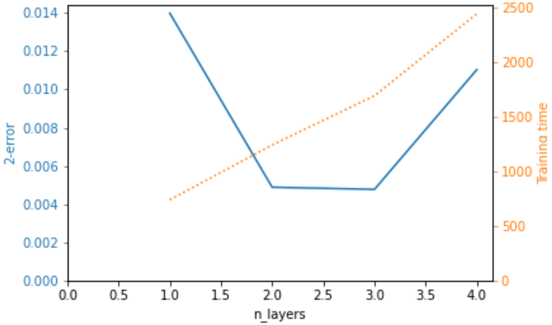
Layers: DGM, BS



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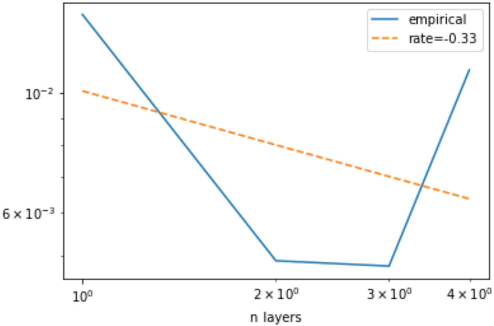
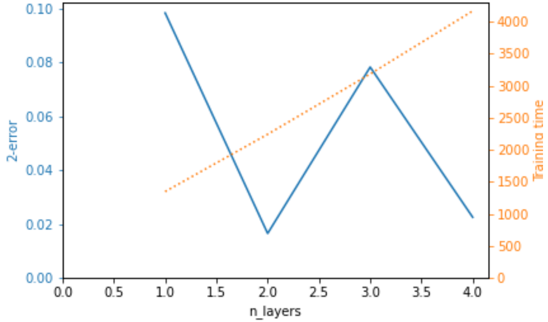


(b) Logarithmic scale

Layers: DGM, Heston



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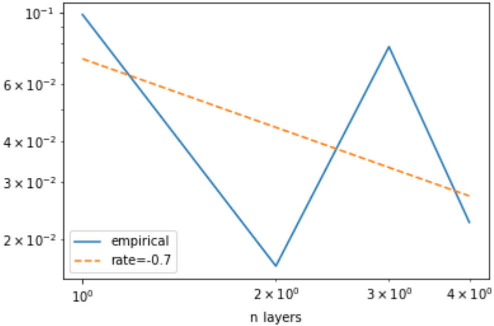
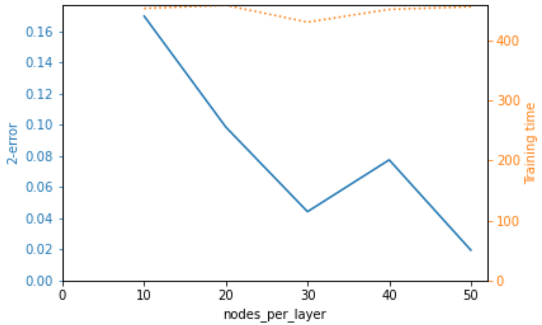


(b) Logarithmic scale

Nodes per layer: TDGF, BS



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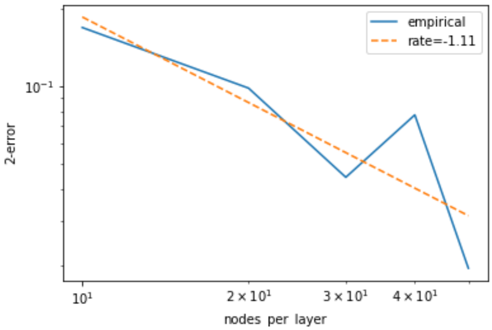

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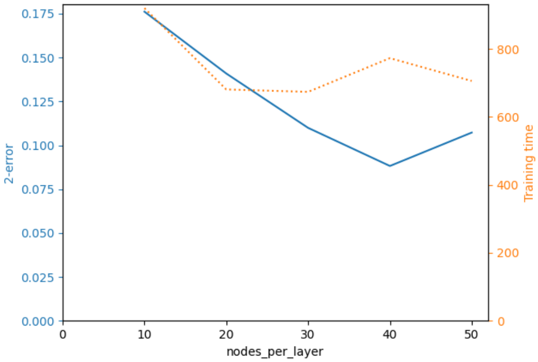

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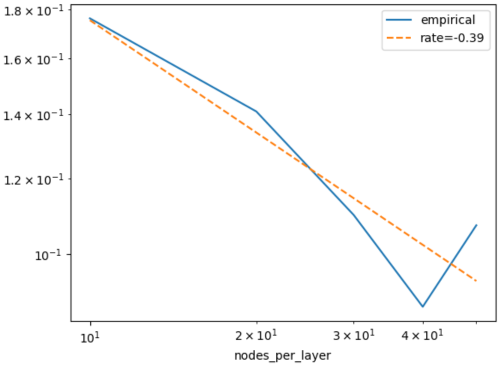
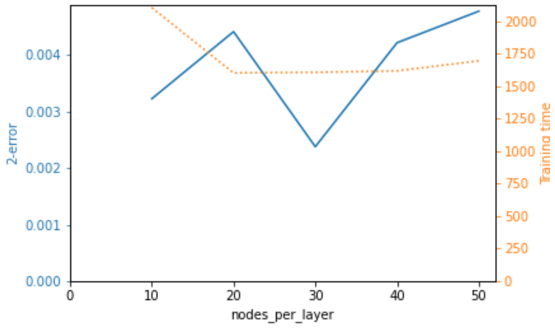
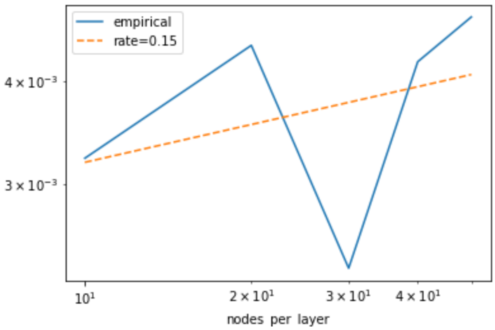


(b) Logarithmic scale

Nodes per layer: DGM, BS

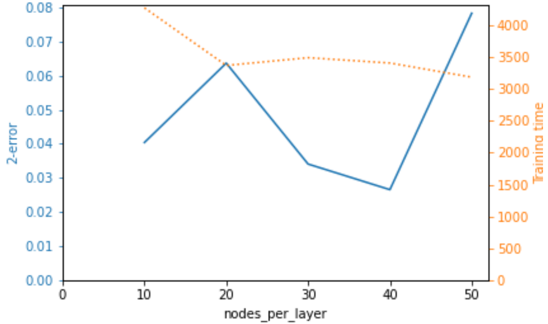


(a) Linear scale

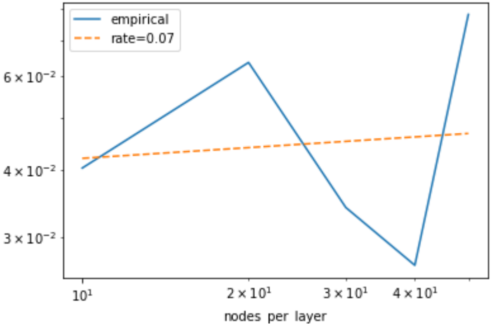


(b) Logarithmic scale

Nodes per layer: DGM, Heston

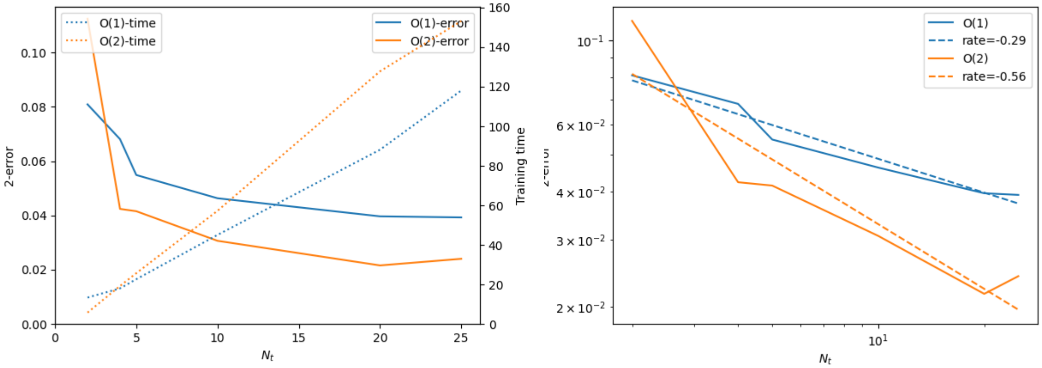


(a) Linear scale



(b) Logarithmic scale

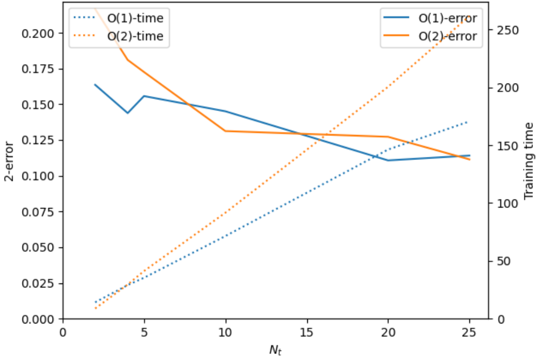
Time steps, BS



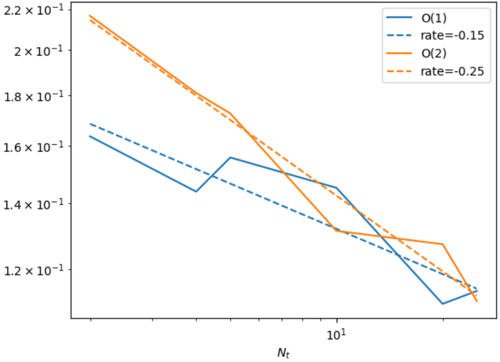
(a) Linear scale

(b) Logarithmic scale

Time steps, BS



(a) Linear scale



(b) Logarithmic scale

Conclusion

Parameter	Accuracy	Training time
Sampling stages	✓	✓
Samples	-	×
Layers	✓	✓
Nodes per layer	-	×
Time steps	✓	✓

Error Analysis of Deep PDE Solvers for Option Pricing

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